Notes on Seiberg-Witten curves

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1 $\mathcal{N}=1$ multiplets and the general Lagrangian

Chiral multiplet The $\mathcal{N}=1$ chiral field Φ is annihilated by the $\bar{D}_{\dot{\alpha}}$ differential operator. Therefore in terms of $y=x+i\theta\sigma\bar{\theta}$, we can expand it as

$$\Phi=\phi(y)+\sqrt{2}\theta\psi(y)+\theta\theta F(y). \tag{1.1}$$

F is an auxiliary field and the only on-shell d.o.f.s are one complex scalar ϕ and one left-handed Weyl fermion ψ. We can certainly consider the conjugate version of Φ, $\bar{\Phi}$, which is annihilated by the differential D_α.

Vector multiplet The vector field is embedded in a superfield V satisfying $V=V^\dagger$. It certainly transforms under the gauge symmetry.

Before we fix the gauge, the field content is quite complicated, however, there is a special gauge which can simplify the expression a lot. This is called the Wess-Zumino gauge, and under this gauge fixing,

$$V=-\theta\sigma^\mu\bar{\theta}A_\mu(x)+i\theta\theta\bar{\theta}\bar{\lambda}(x)-i\bar{\theta}\bar{\theta}\theta\lambda(x)+\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x). \tag{1.2}$$

A useful superfield is constructed from V by performing D for several times.

$$W_\alpha=-\frac{1}{4}\bar{D}\bar{D}D_\alpha V, \tag{1.3}$$

$$\bar{W}_{\dot{\alpha}}=-\frac{1}{4}DD\bar{D}_{\dot{\alpha}}V. \tag{1.4}$$

The only one important fact we need is that

$$W^\alpha W_\alpha|_{\theta\theta}=-2i\lambda\sigma^\mu\partial_\mu\bar{\lambda}-\frac{1}{2}F^{\mu\nu}F_{\mu\nu}-\frac{1}{2}F^{\mu\nu}\tilde{F}_{\mu\nu}, \tag{1.5}$$

where $\tilde{F}_{\mu\nu}=-\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$.

The Lagrangian density for a pure super gauge theory can thus be written as

$$\mathcal{L}=\frac{1}{4}W^\alpha W_\alpha\Big|_{\theta\theta}+\frac{1}{4}\bar{W}^\alpha\bar{W}_\alpha\Big|_{\bar{\theta}\bar{\theta}}. \tag{1.6}$$

Interaction and Lagrangian We can couple the chiral multiplet to some Abelian/non-Abelian gauge field V. The invariant action is known as

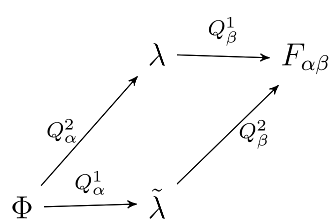
$$\int \mathrm{d}^4\theta \bar{\Phi}^J e^{V_a (T^a)^I{}_J} \Phi_I. \tag{1.7}$$

This term corresponds to the kinetic term of the chiral fields and we replaced the usual derivative to a covariant derivative. There is also an interaction term among different chiral multiplets. It is encoded in the superpotential $W(\Phi)$, and we have the following Lagrangian density,

$$\int \mathrm{d}^2\theta W(\Phi) + \mathrm{c.c.} \tag{1.8}$$

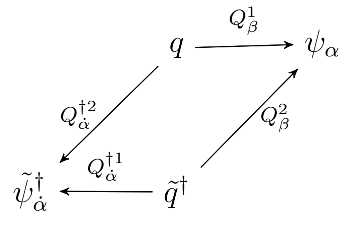
2 $\mathcal{N} = 2$ multiplets

The $\mathcal{N} = 2$ vector multiplet is constituted from one $\mathcal{N} = 1$ chiral multiplet and one $\mathcal{N} = 1$ vector multiplet. Note that the chiral multiplet can be generated by $Q_{1\alpha}$, i.e. start from one highest weight state $|\Phi\rangle$, we obtain the Weyl fermions $|\tilde{\lambda}_\alpha\rangle = Q_{1\alpha} |\Phi\rangle$. Given that the highest weight state is annihilated by $\bar{Q}$'s, further action of $\bar{Q}$'s will only give derivatives of the highest-weight scalar field Φ. $Q_1 Q_2 |\Phi\rangle$ can be identified with the auxiliary field. Similarly, starting from a Weyl fermion λ_α, we act $Q_{1\beta}$ on it, obtaining $\sigma^{\mu\nu}{}_{\alpha\beta} F_{\mu\nu}$. We can schematically write this transformation as $|F_{\alpha\beta}\rangle = Q_\beta |\lambda_\alpha\rangle$. Thus the $\mathcal{N} = 2$ vector multiplet can be summarized into the following diagram.



The chiral multiplet must be in the adjoint representation of the gauge group.

In the same spirit, for hypermultiplet, we combine two $\mathcal{N} = 1$ chiral multiplets, (q, ψ_α) and $(\tilde{q}, \tilde{\psi}_\alpha)$.



These two $\mathcal{N} = 1$ multiplets are respectively in the representation of the flavor symmetry group R and its complex conjugate representation $R^\dagger$.

Example: moment map When we consider the hypermultiplets in the representation of SU(N), we introduce the following notation,

$$(Q^I_a) = \begin{pmatrix} q_a \\ \tilde{q}^\dagger_a \end{pmatrix}, \quad (\tilde{Q}^{Ia}) = \begin{pmatrix} \tilde{q}^a \\ -q^{\dagger a} \end{pmatrix}. \tag{2.1}$$

We can construct an adjoint quantity of the SU(N) group with two $\mathrm{SU}(2)_R$ indices,

$$(\mu^{IJ}) = Q^I_a \tilde{Q}^{Jb} - \frac{\delta_a{}^b}{N} Q^I_c \tilde{Q}^{Jc}. \tag{2.2}$$

One component of this matrix, say μ^{11}, corresponds to the primary state in the shorten multiplet $\hat{B}_1$.

3 $\mathcal{N} = 2$ general action

The $\mathcal{N} = 2$ Lagrangian can be obtained by pasting two $\mathcal{N} = 1$ actions and imposing the $\mathrm{SU}(2)_R$ symmetry. The Lagrangian for the vector multiplet is given by

$$-\frac{i\tau}{8\pi}\int \mathrm{d}^2\theta \mathrm{tr} W_\alpha W^\alpha + \frac{\mathrm{Im}\tau}{4\pi}\int \mathrm{d}^4\theta \mathrm{tr}\Phi^\dagger e^{[V,\cdot]}\Phi + \mathrm{c.c.} \tag{3.1}$$

In fact, the convention of normalization used here is not convenient when we try to discuss the deformation of the theory. We can go to the canonical normalization with the rescaling $W \to gW$, where g is the gauge coupling, and we obtain

$$-\frac{i\tau}{8\pi}\int \mathrm{d}^2\theta \mathrm{tr} W_\alpha W^\alpha + \frac{\mathrm{Im}\tau}{4\pi}\int \mathrm{d}^4\theta \mathrm{tr}\Phi^\dagger e^{[gV,\cdot]}\Phi + \mathrm{c.c.} \tag{3.2}$$

Now we see that we can add the exactly deformation FF or $F\tilde{F}$ to deform the Lagrangian without breaking the supersymmetry or any other symmetries. By redefining the vector field F back to the canonical normalization, we are able to shift the gauge coupling g in other part of the action and the θ angle.

The matter part can be added when we prepare two $\mathcal{N} = 1$ chiral multiplets, Q and $\tilde{Q}$. The Lagrangian reads

$$\int \mathrm{d}^4\theta (Q^\dagger e^{gV} Q + \tilde{Q} e^{-gV} \tilde{Q}^\dagger) + \int \mathrm{d}^2\theta (\tilde{Q}\Phi Q + \mathrm{c.c.}) + \int \mathrm{d}^2\theta (\mu \tilde{Q} Q + \mathrm{c.c.}). \tag{3.3}$$

Again we can see that the exactly marginal deformation we obtained before keeps working even after we include matters. This fact leads to an important consequence: this exactly marginal deformation for $\mathcal{N} = 2$ theories is also exactly marginal for $\mathcal{N} = 4$ theories. Let us have a further look at the $\mathcal{N} = 4$ Lagrangian.

$\mathcal{N}=4$ **Lagrangian** The only $\mathcal{N}=4$ multiplet is a vector multiplet, which is consisted of an $\mathcal{N}=2$ vector multiplet and an $\mathcal{N}=2$ hypermultiplet (with its conjugate contained too), and its Lagrangian is straightforward to be written down.

$$-\frac{i\tau}{8\pi}\int \mathrm{d}^2\theta \mathrm{tr} W_\alpha W^\alpha + \frac{\mathrm{Im}\tau}{4\pi}\int \mathrm{d}^4\theta \mathrm{tr}\Phi^\dagger e^{[V,\cdot]}\Phi + \frac{\mathrm{Im}\tau}{4\pi}\int \mathrm{d}^4\theta (Z^\dagger e^{g[V,\cdot]}Z + \tilde{Z}e^{-g[V,\cdot]}\tilde{Z}^\dagger)$$
$$+\frac{\mathrm{Im}\tau}{4\pi}\int \mathrm{d}^2\theta \tilde{Z}[\Phi, Z] + \mathrm{c.c.} \qquad (3.4)$$

Since the exactly marginal deformation FF and $F\tilde{F}$ preserve the $\mathrm{SU}(4)_R$ symmetry, we know that they are also the exactly marginal deformation of the $\mathcal{N}=4$ SYM. There is also a mass deformation of the theory by adding a mass term

$$\int \mathrm{d}^2\theta(\mu\tilde{Z}Z + \mathrm{c.c.}), \qquad (3.5)$$

and the resulting theory is called an $\mathcal{N}=2^*$ theory.

4 $\mathcal{N}=2$ effective action

The above Lagrangian is obtain by requiring the theory to be renormalizable. However, for a Wilson-type effective action, we do not need this condition and the most general action written in terms of $\mathcal{N}=2$ superfields, $\Phi_{\mathcal{N}=2}(x,\theta_1,\bar{\theta}^1,\theta_2,\bar{\theta}^2) = \phi(x) + \theta_i\psi^i + \bar{\theta}_i\bar{\chi}^i + \cdots + \theta_1^2\bar{(}\theta^1)^2\theta_2^2\bar{(}\theta^2)^2 D'$, as

$$S = \int \mathrm{d}^4x\mathrm{d}^2\theta_1\mathrm{d}^2\theta_2\mathrm{tr}\mathcal{F}(\Phi_{\mathcal{N}=2}) + \mathrm{c.c.} \qquad (4.1)$$

The $\mathcal{N}=2$ vector multiplet can be expressed with the $\mathcal{N}=2$ chiral superfield as

$$\Phi_{\mathcal{N}=2}(y,\theta) = \Phi(y,\theta_1) + \sqrt{2}\theta^{2\alpha}W_\alpha(y,\theta_1) + (\theta^1)^2 G(y,\theta_1), \qquad (4.2)$$

when we are considering an abelian effective gauge theory, the effective action can be integrated with respect to θ_2 and we obtain

$$S = \int \mathrm{d}^4x\mathrm{d}^2\theta^1\left(-\frac{1}{2}\frac{\partial\mathcal{F}}{\partial\Phi^a}G^a - \frac{\partial^2\mathcal{F}}{\partial\Phi^a\Phi^b}W^{a\alpha}W^b_\alpha + \mathrm{c.c.}\right). \qquad (4.3)$$

$\mathcal{F}$ is called the prepotential and it is known that to realize the gauge theory, we have to set $G = -\frac{1}{2}\int \mathrm{d}^2\bar{\theta}_1\bar{\Phi}^\dagger e^{2gV}$.

Matter can also couple to the theory with the usual matter Langrangian added.

5 Seiberg-Witten theory for SU(2) pure gauge theory and with matter

The Seiberg-Witten theory is an algorithm to compute the prepotential of the effective theory in the Coulomb branch of $\mathcal{N}=2$ super Yang-Mills theory.

First, let us give a brief instruction on what a Coulomb branch and what a Higgs branch is. In the original normalizable theory, we have the vacuum potential (variation with respect to the auxiliary field D)

$$\frac{1}{g^2}\left[\Phi^\dagger, \Phi\right] + \left(Q_i Q^{\dagger i} - \tilde{Q}_i^\dagger \tilde{Q}^i\right)|_{\text{traceless}} = 0. \tag{5.1}$$

The Coulomb branch is where only the VEV of the scalar in Φ is nonzero, and the Higgs branch is where only the scalars in hypermultiplets are nonzero. We can also have all of them to be non-vanishing, and that is generally called a mixed branch.

At a generic point of the Coulomb branch, the gauge group breaks down to U(1)'s. We introduce a Riemann surface with branch cuts, whose coordinates are given by the vev of $u_i = rmtr(\Phi^i)$ for $i = 2, \ldots, n$, where n is the rank of the gauge group. The prepotential, which can be expressed as a function of u^i's, is considered to be defined on this Riemann surface $\mathcal{C}$. In the high energy limit, we can compute the non-abelian gauge theory perturbatively. That is to say, we can easily obtain the boundary condition for $\mathcal{F}$ with large $|u|$ (high energy limit). In the effective theory, we have the BPS bound

$$Z = n^i a_i + m_i a_D^i + \mu_j f_j, \tag{5.2}$$

where a_i is the diagonal entries of the vev of the scalar field Φ in the vector multiplet, $a_D^i := \frac{\partial \mathcal{F}}{\partial a_i}$, and μ_j is the mass of hypermultiplets with flavor charge f_j. We also know from the effective action that the effective gauge coupling is given by

$$\tau_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} = \frac{\partial a_D^j}{\partial a_i}. \tag{5.3}$$

The Seiberg-Witten theory gives us the prescription to compute a and a_D and by integrating a_D over a, we can easily obtain the expression of $\mathcal{F}$.

The first known example is the pure SU(2) gauge theory, whose Riemann surface $\mathcal{C}$ is specified by a torus form

$$\Lambda^2 z + \frac{\Lambda^2}{z} = x^2 - u. \tag{5.4}$$

Since it is an SU(2) theory, $u = \frac{1}{2}\langle \text{tr}\Phi^2 \rangle \sim a^2$. We have four branch points of the Riemann surface, respectively 0, ∞ and $z_\pm = -\frac{u}{2\Lambda^2} \pm \sqrt{\frac{u^2}{4\Lambda^4} - 1}$. The Riemann surface is two $\hat{\mathbb{C}}$ connected by the two branch cuts running from 0 to z_- and from ∞ to z_+. It is topologically equivalent to a torus, and we define two fundamental circles, one penetrating the branch cuts denoted as B-cycle, and another denoted as A-cycle. The Seiberg-Witten prescription states that we can compute a and a_D with the Seiberg-Witten differential $\lambda = \frac{x}{z}\mathrm{d}z$,

$$a = \frac{1}{2\pi i}\oint_A \lambda, \quad a_D = \frac{1}{2\pi i}\oint_B \lambda. \tag{5.5}$$

We can check whether the monodromy of the perturbative theory agrees with that in the Seiberg-Witten theory. When we break the gauge group from SU(2) to U(1)'s, we have

$$F_{\mu\nu}^{SU(2)} = \mathrm{diag}\left(F_{\mu\nu}^{U(1)}, -F_{\mu\nu}^{U(1)}\right), \tag{5.6}$$

and the U(1) gauge coupling is related to the SU(2)'s by $\tau^{U(1)} = 2\tau^{SU(2)}$. Thus from the SU(2) perturbative calculation, we have

$$\tau^{U(1)}(a) = -\frac{8}{2\pi i}\log\frac{a}{\Lambda} + \dots, \tag{5.7}$$

where $\dots$ expresses only the non-perturbative effects, since $\mathcal{N} = 2$ theories are one-loop exact. Apparently, even in the perturbative region, this function is not simple-valued, and we have a monodromy around the infinity

$$M_\infty = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}, \tag{5.8}$$

note that $a_D = -\frac{8a}{2\pi i}\log\frac{a}{\Lambda} + \dots$ and by going around the infiniy as $u \sim a^2$ gains a phase 2π, $a \to ae^{\pi i}$. In the Seiberg-Witten theory, around $u \sim \infty$ (i.e. $u \gg \Lambda^2$), we have $x \sim \sqrt{u}$, [1] and $z_\pm \sim -(u/\Lambda^2)^\pm$. Since z_+ is very small, we can deform the contour to locate at $|z| = 1$ and we can easily see that $a = \sqrt{u}$. We have to be careful when we try to compute a_D with $x \sim \sqrt{u}$, because it only holds up to $z \sim u$ and down to $z \sim 1/u$. We can only compute using the approximated value of x along the line from z_+ to z_+, and we obtain

$$a_D \simeq 2\frac{\sqrt{u}}{2\pi i}\int_{z_+}^{z_-}\frac{\mathrm{d}z}{z} = \frac{2\sqrt{u}}{\pi i}\log\frac{u}{\Lambda^2}. \tag{5.9}$$

For a, we can deform the integral contour (refer to Figure 1) to, say, the unit circle $|z| = 1$ to obtain

$$a \simeq \frac{\sqrt{u}}{2\pi i}\oint\frac{1}{z}\mathrm{d}z = \sqrt{u}. \tag{5.10}$$

These expressions for a and a_D agree with the perturbative calculation results. The prepotential $\mathcal{F}$ can be found by integrating a_D over a to obtain

$$\mathcal{F} \simeq \frac{a^2}{\pi i} - \frac{2a^2}{\pi i}\log\frac{a}{\Lambda}, \tag{5.11}$$

in the perturbative region.

We note that there are two special points on the u-plane, with $u = \pm 2\Lambda^2$, where we have a_D vanish at these points. That means instead of massless quarks in the perturbative regime, we now have massless monopoles ($a_D = 0$) at such strong coupling points.

[1] Note that the branch cut is for function of z, so we have to restrict us to one branch of u. Fortunately, the overall sign does not matter in the BPS bound.

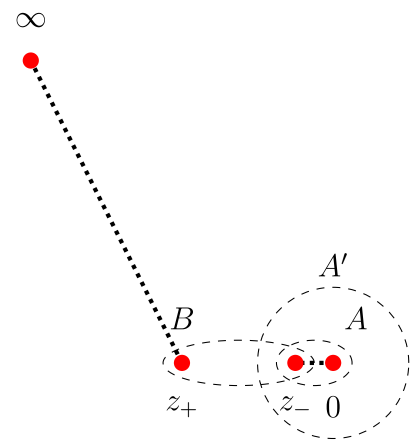


Figure 1: Two branch cuts and A, B-cycles together with the deformed A-cycle, A′-cycle.

Now let us turn to the matter case, $N_f = 1$. The curve is known as

$$\frac{2\Lambda(x-\mu)}{z} + \Lambda^2 z = x^2 - u, \tag{5.12}$$

where μ is the mass of additional matter. Now we have three branch points on the z-plane, and in the limit $u \to \infty$, we have two almost degenerate points at $z_+^{1,2} = \pm i\frac{\Lambda}{\sqrt{u}}$, and one around the infinity, $z = -\frac{u}{\Lambda^2}$. Again, in such a perturbative region, we have $a \sim \sqrt{u}$ and

$$a_D \sim -2\frac{\sqrt{u}}{2\pi i}\int_{\Lambda/\sqrt{u}}^{u/\Lambda^2} \frac{\mathrm{d}z}{z} = -\frac{3a}{\pi i}\log a. \tag{5.13}$$

We note that in the last equation, we dropped out some subleading contribution with order a. In general, we have the one-loop coupling running in the perturbative region

$$\tau(a) = -\frac{2(4-N_f)}{2\pi i}\log\frac{a}{\Lambda}, \tag{5.14}$$

and we see that the curve we are considering reproduces this result for $N_f = 1$.

When we go to the case of $N_f = 2$, we can further put one more factor of the form $(x-\mu)$ on the numerator of $1/z$.

$$\frac{(x-\mu_1)(x-\mu_2)}{z} + 4\Lambda^2 z = x^2 - u. \tag{5.15}$$

It can be checked that this is indeed the Seiberg-Witten curve for $N_f = 2$, and in fact we can change the variable $z \to z' = \frac{(x-\mu_2)z}{(2\Lambda)}$ to make the two factors of $(x-\mu)$ in a more symmetric form,

$$\frac{2\Lambda(x-\mu_1)}{z} + 2\Lambda(x-\mu_2) = x^2 - u. \tag{5.16}$$

Certainly, the Seiberg-Witten differential does not change its form.

In the same way, for $N_f = 3$, we can write down the curve as

$$\frac{(x-\mu_1)(x-\mu_2)}{z} + 2\Lambda(x-\mu_3)z = x^2 - u, \tag{5.17}$$

and in the case of $\mathcal{N}_f = 4$,

$$c_1\frac{(x-\mu_1)(x-\mu_2)}{z} + c_2(x-\mu_3)(x-\mu_4)z = x^2 - u, \tag{5.18}$$

with two complex constants $c_{1,2}$. Interesting things happen in this case, as we can see when we set u to be very large, we can evaluate x to

$$x \sim \sqrt{\frac{u}{\frac{c_1}{z} + c_2 z - 1}}, \tag{5.19}$$

and the location of branches points no longer depends on u, which leads to the conclusion that

$$a_D \propto a, \tag{5.20}$$

and we see that the gauge coupling τ is not running.

6 Seiberg-Witten curve from brane construction

In arXiv:hep-th/9703166, Witten contructed a large class of 4d $\mathcal{N} = 2$ gauge theories in the D4-NS5 system of type IIA superstring and analyzed them in the M-theory.

The illustration is very simple. We have NS5-branes stretching along $x^{0,1,2,3,4,5}$ and D4's stretching along $x^{0,1,2,3,6}$. We set all NS5's locating at $x^{7,8,9} = 0$, but separated in the 6-th direction.

We know that under the dragging effect of D4-branes, x^6 of NS5-branes takes the form

$$x^6 = k \log|v| + \text{const.}, \tag{6.1}$$

where $v = x^4 + ix^5$ and the D4 under consideration is now put at the origin of v-plane and k is some constant related to the tension of branes. With only one D4 ending on the NS5 brane, we see that x^6 diverges at the the infinity of v-plane, to cancel this divergence, we need equal numbers of D4's on two sides, i.e.

$$x^6 = k\sum_{i=1}^{N} \log|v - a_i| - k\sum_{i=1}^{N} \log|v - b_i| + \text{const.} \tag{6.2}$$

From the convergence of action of the NS5-brane, we also have the constraint

$$\sum_i a_i - \sum_i b_i = 0. \tag{6.3}$$

This configuration generally gives the A-type quiver gauge theories, where strings extended cross the NS5 and ended on two neighboring D4's give the hypermultiplet taking bifundamental representation of the respective gauge group rising from two D4's. The gauge coupling of the vector multiplet on each D4 reads

$$\frac{1}{g_{\alpha}^{2}(v)}=\frac{x_{\alpha}^{6}(v)-x_{\alpha-1}^{6}(v)}{\lambda}, \tag{6.4}$$

where α denotes that the gauge coupling belongs to the brane between the $(\alpha-1)$-th and the α-th NS5-brane. We clearly see a logarithmic divergence at small v and this can be interpreted as the UV behavior of the one-loop correction of gauge couplings.

Witten further lifted this configuration up to the M-theory by adding one $x^{1}0$-circle, and we identify the complexified gauge coupling as

$$-i\tau_{\alpha}(v)=s_{\alpha}(v)-s_{\alpha-1}(v), \tag{6.5}$$

with $s=\frac{1}{R}(x^{6}+ix^{10})$. Without considering the balance of force, we will have a running of the gauge coupling of the form

$$-(2N_c-N_f)\ln v, \tag{6.6}$$

which is a well-known form of the beta function of 4d SYMs. We see that the brane construction always gives gauge theories without gauge coupling running at the IR region of the string theory, i.e. SCFTs.

Lifting up to M-theory, all these branes become M5-branes. D4 is obtained by wrapping on the M-theory circle S^{1} in the x^{10} direction. Therefore, the 4d theory we are considering is deduced from the M5-brane on the manifold $\mathbb{R}^{1,3}\times\Sigma_{6,10}$. The self-dual tensor in the tensor multiplet of M5-brane can be reduced as

$$T=F\wedge\Lambda+*F\wedge*\Lambda, \tag{6.7}$$

where Λ is harmonic one-form ($\mathrm{d}\Lambda=\mathrm{d}*\Lambda=0$) on Σ. For a genus g surface Σ, we will obtain g different types of gauge fields and the low energy effective theory is believed to have gauge group $\mathrm{U}(1)^{g}$.

Let us consider how to obtain the Seiberg-Witten curve from this brane construction. In the language of M-theory, we use the coordinate $t=\exp(-s)$ and we would like to describe the brane "curve" on the t-v plane as

$$F(t,v)=0. \tag{6.8}$$

The simplest set up is with two NS5-branes and k D4-branes stretched between them. In the classical picture, we expect that there are k roots of v by solving the above algebraic equation and two roots of t. That is to say the curve takes the form

$$A(v)t^{2}+B(v)t+C(v)=0. \tag{6.9}$$

The zeroes of $C(v)$ can be identified as the positions of external D4-branes, as at these places we always have one solution $t = 0$, $x^6 \to \infty$, which agrees with the near center behavior (6.2). Similarly, the zeroes of $A(v)$ correspond to where $t \to \infty$, i.e. $x^6 \to -\infty$, the places of external D4's attached from the left-hand side.

Therefore, we reached the conclusion that for a pure SU(k) gauge theory, the curve has no zeroes in $A(v)$ and $C(v)$ and thus reads

$$t^2 + B(v)t + 1 = 0, \tag{6.10}$$

or

$$t^2 = \frac{B(v)^2}{4} - 1, \tag{6.11}$$

after shift and rescaling of t and v. The form of $B(v)$ is

$$B(v) = v^k + \sum_{i=2}^{k} u_i v^{k-i}. \tag{6.12}$$

When t is very large, we recover the asymptotic behavior of the bending effect $t \propto v^k$, (6.2) and also for small t, $t \propto v^{-k}$.

Note that in the standard form of the SW curve, t corresponds to z and v is x. To add hypermultiplets into the theory, we can either add D4-branes from the left-hand side or from the right-hand side. This exactly claims that in the SW curve, we can either put the factor $(z - m)$ in front of z or above $1/z$. We note that transferring all matters from left to right can be done by a parity transformation, reversing the x^6 direction, which corresponds to $t \to t^{-1}$ and is a familiar transformation to us. However, pulling only part of the external branes to the other end cannot be realized by coordinate transformation and thus can not be realized in the SL(2,$\mathbb{Z}$) transformation. This is a highly non-trivial symmetry of the system, while trivialized under the brane picture.

More generally for quiver gauge theories with n NS5-branes, the curve can be written down as

$$A_0(v)t^n + A_1(v)t^{n-1} + \cdots + A_n(v) = 0. \tag{6.13}$$

Again, external D4's correspond to zeroes in $A_n(v)$ and $A_0(v)$ respectively. For SU(2) quiver gauge, $A_i(v) = v^2 - v_i$, and we recover the general form of the SW curve claimed before,

$$\prod_i (t - t_i)v^2 = tU_{n-1}(t), \tag{6.14}$$

where U_{n-1} is a polynomial of degree $n - 1$.

7 Curve for $\mathcal{N} = 4$ SYM

The curve for $\mathcal{N} = 4$ is vague, since we cannot directly read it off from the usual brane construction lifted to M-theory, which definitely breaks the supersymmetry down to $\mathcal{N} = 2$. One way to obtain such a theory in the string theory is to consider N D3-branes in type IIB. The 4d theory living on D3-branes is a U(N) SYM theory. The gauge coupling is determined by the tension of the D3-rabnes. We can compactify two directions of the transverse space on a torus, and take T-duality along on of the circles. The resulting theory is with N D4-branes on $\mathbb{R}^{1,3} \times S^1$, i.e. the low energy theory is an $\mathcal{N} = 2$ 5d SYM compactified on S^1. This setup resembles the brane construction for general 4d $\mathcal{N} = 2$ quiver gauge theories, which will be described in details in later sections, and now we have no NS5-branes, that is to say, there is no t in the curve (in Witten's terminology). Therefore, we expect the curve to take the form,

$$\prod_{i=1}^{N}(v - v_i) = 0. \tag{7.1}$$

8 Derivation of Prepotential from Partition Function

The Seiberg-Witten curve is extracted out from the Nekrasov partition function by taking the limit $\epsilon_{1,2} \to 0$.

An important identity to be used in the derivation is given by

$$\frac{\Delta(0)\Delta(\epsilon_1+\epsilon_2)}{\Delta(\epsilon_1)\Delta(\epsilon_2)} = \exp\left(\epsilon_1\epsilon_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2}\log\Delta(t)\bigg|_{t=0} + \mathcal{O}(\epsilon^2)\right). \tag{8.1}$$

The k-instanton partition function of 5d U(N) gauge theory is given by

$$Z_k = \frac{1}{k!}\left(\frac{[\epsilon_1+\epsilon_2]}{[\epsilon_1][\epsilon_2]}\right)^k \oint \left(\prod_{i=1}^{k}\frac{\mathrm{d}\phi_i}{2\pi i}\right)\prod_{i=1}^{k}\prod_{a=1}^{N}\frac{1}{[\mathfrak{a}_a - \phi_i][\phi_i - \mathfrak{a}_a + \epsilon_1+\epsilon_2]}\prod_{\substack{i,j=1\\ i\neq j}}^{k} S^{-1}(\phi_i - \phi_j), \tag{8.2}$$

where

$$S(x) := \frac{[x+\epsilon_1][x+\epsilon_2]}{[x][x+\epsilon_1+\epsilon_2]}. \tag{8.3}$$

One can replace $[x]$ by x in the above expression to take the 4d limit. Note that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log[x-t] = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\log\left(2\sinh\left(\frac{x-t}{2}\right)\right) = -\frac{1}{[x-t]^2}, \tag{8.4}$$

and by introducing the density

$$\rho(x) = \epsilon_1\epsilon_2\sum_{i=1}^{k}\delta(x-\phi_i), \tag{8.5}$$

we have

$$Z_k \sim \exp\left(-\frac{\mathcal{E}}{\epsilon_1\epsilon_2}\right), \tag{8.6}$$

with the free energy given by

$$\mathcal{E} = \int \mathrm{d}x\mathrm{d}y \frac{\rho(x)\rho(y)}{[x-y]^2} + \sum_{a=1}^{N} \int \mathrm{d}x \rho(x) \left(\log[\mathfrak{a}_a - x][x - \mathfrak{a}_a]\right). \tag{8.7}$$

In the partition function, we need to consider the weighted sum of all integer k, [2]

$$Z_{U(N)} = \sum_{k=0}^{\infty} \Lambda^{2Nk} Z_k, \tag{8.8}$$

and the instanton counting parameter Λ can be absorbed into the free energy $\mathcal{E}$, to define

$$\mathcal{E}_\Lambda = \int \mathrm{d}x\mathrm{d}y \frac{\rho(x)\rho(y)}{[x-y]^2} + \sum_{a=1}^{N} \int \mathrm{d}x \rho(x) \left(\log\left(\frac{[\mathfrak{a}_a - x][x - \mathfrak{a}_a]}{\Lambda^2}\right)\right) \tag{8.9}$$

Let us introduce the following kernel function,

$$k_\Lambda(x) := \frac{x^2}{2}\left(\log\frac{|x|}{\Lambda} - \frac{3}{2}\right), \tag{8.10}$$

satisfying

$$k''_\Lambda(x) = \log\left(\frac{|x|}{\Lambda}\right), \quad k^{(4)}_\Lambda(x) = -\frac{1}{x^2}. \tag{8.11}$$

Then we can rewrite $\mathcal{E}_\Lambda$ in the 4d limit to

$$\mathcal{E}^{4d}_\Lambda = -\int \mathrm{d}x\mathrm{d}y \rho''(x)\rho''(y) k_\Lambda(x-y) + \sum_{a=1}^{N} \int \mathrm{d}x \left(\rho''(x) k_\Lambda(x - \mathfrak{a}_a) + \rho''(x) k_\Lambda(\mathfrak{a}_a - x)\right). \tag{8.12}$$

It is natural to define a 5d version of the kernel function, k^{5d}_Λ, to lift the above expression to to 5d. We further note that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}|x-a| = 2\delta(x-a), \tag{8.13}$$

we can define

$$f(x) = \rho(x) - \frac{1}{2}\sum_a |x - \mathfrak{a}_a|, \tag{8.14}$$

[2]Of course, we "forgot" to consider the contribution from $\left(\frac{[\epsilon_1+\epsilon_2]}{[\epsilon_1][\epsilon_2]}\right)^k$, as it can be absorbed into the gauge coupling $\mathfrak{q} = \Lambda^{2N}$.

to rewrite

$$\mathcal{E}_\Lambda^{4d} = -\int \mathrm{d}x\mathrm{d}y f''(x)f''(y)k_\Lambda(x-y) + \sum_{a,b=1}^{N} k_\Lambda(\mathfrak{a}_a - \mathfrak{a}_b). \tag{8.15}$$

The above free energy only covers the non-perturbative part, but we also need to compensate with the perturbative part. The perturbative part (involving the classical piece and the one-loop factor) is described by the function,

$$\gamma_{\epsilon_1,\epsilon_2}(x|R,\Lambda) = \frac{1}{2\epsilon_1\epsilon_2}\left(-\frac{R}{6}\left(x+\frac{\epsilon_1+\epsilon_2}{2}\right)^3 + x^2\log(R\Lambda)\right) + \sum_{k=1}^{\infty}\frac{1}{k}\frac{e^{-Rnx}}{(e^{Rn\epsilon_1}-1)(e^{Rn\epsilon_2}-1)}, \tag{8.16}$$

where R is the radius of S^1 on which the 5d theory is compactified, and it is usually absorbed into $\epsilon_{1,2}$, Λ and $\mathfrak{a}_a$'s (when R does not appear explicitly, it is absorbed). The perturbative partition function is given by

$$Z_{U(N)}^{pert}(\{\mathfrak{a}_a\};R) = \Lambda^{\frac{1-N^2}{12}}\exp\left(\sum_{\alpha:\ \mathrm{root}}\gamma_{\epsilon_1,\epsilon_2}(\mathfrak{a}_\alpha|R,\Lambda)\right). \tag{8.17}$$

For our purpose, we take the unrefined limit $\epsilon_1 = -\epsilon_2 = \hbar$, and expand $\gamma_{\hbar,-\hbar}$ as

$$\gamma_{\hbar,-\hbar}(x|R,\Lambda) = \sum_{g=0}^{\infty}\gamma_g(x|R,\Lambda)\hbar^{2g-2}, \tag{8.18}$$

and we have

$$\gamma_0 = \frac{x^3}{12} - \frac{x^2}{2}\log(R\Lambda) - \frac{1}{R^2}\mathrm{Li}_3(e^{-Rx}). \tag{8.19}$$

Summing over all $\gamma_{\epsilon_1,\epsilon_2}(\mathfrak{a}_\alpha|R,\Lambda)$ and taking the 4d limit ($R\to 0$?), it in fact reduces to

$$\gamma_0^{4d} = \frac{x^2}{2}\log\left(\frac{x}{\Lambda}\right) - \frac{3x^2}{4}, \tag{8.20}$$

and $\gamma_{\epsilon_1,\epsilon_2}^{4d}$ can be determined from

$$\gamma_{\epsilon_1,\epsilon_2}^{4d}(x;\Lambda) = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\Lambda^s}{\Gamma(s)}\int_0^\infty \mathrm{d}t\ t^{s-1}\frac{e^{-tx}}{(e^{\epsilon_1 t}-1)(e^{\epsilon_1 t}-1)}\right)\Bigg|_{s=0}. \tag{8.21}$$

For example, at the leading order of $\epsilon_1 = -\epsilon_2 = \hbar$,

$$\begin{aligned}\gamma_0^{4d} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\Lambda^s}{\Gamma(s)}\int_0^\infty \mathrm{d}t\ t^{s-3}e^{-tx}\right)\Bigg|_{s=0} &= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\Lambda^s}{\Gamma(s)}x^{2-s}\Gamma(s-2)\right)\Bigg|_{s=0}\\ &= \frac{x^2}{2}\log\left(\frac{x}{\Lambda}\right) - \frac{3}{4}x^2,\end{aligned} \tag{8.22}$$

where we used that

$$\lim_{s\to 0}\frac{\Gamma(s-2)}{\Gamma(s)} = \frac{1}{2}, \quad \lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\Gamma(s-2)}{\Gamma(s)}\right) = \frac{3}{2}. \tag{8.23}$$

We note that γ_0^{4d} coincides with the kernel function $k_\Lambda(x)$, but contributes to the free energy with an opposite sign, so at the end the full prepotential is described by

$$\mathcal{F} = \mathcal{F}^{pert} + \mathcal{F}^{inst} = -\frac{1}{\hbar^2}\int \mathrm{d}x\mathrm{d}y f''(x)f''(y)k_\Lambda(x-y). \tag{8.24}$$

9 Derivation of Seiberg-Witten Curve from Prepotential

We want to maximize the above prepotential to find the saddle point of the partition function, so that the thermodynamic limit of the partition function is completely characterized by this critical prepotential.

As the Coulomb branch parameters are fixed in the maximization process, we can introduce Lagrange multipliers, ξ_1, ξ_2, ..., ξ_N to instead minimize

$$\mathcal{S}_\Lambda(f) = -\mathcal{F} - 4\sum_{a=1}^{N} \xi_a \mathfrak{a}_a. \tag{9.1}$$

There in fact exists a concave and piecewise-linear function, ${}^\exists\sigma(x)$, defined on $x \in [-N, N]$, satisfying

$$\sum_a \xi_a \mathfrak{a}_a = -\frac{1}{2}\int_{\mathbb{R}} \mathrm{d}x \sigma\left(f'(x)\right), \tag{9.2}$$

and

$$\sigma'(x) = \xi_i, \quad \text{for } x \in [-N + 2(i-1), -N + 2i], \tag{9.3}$$

$$\sigma(-N) = -\sigma(N) = -\sum_a \xi_a. \tag{9.4}$$

Using this so-called surface tension function $\sigma(x)$, we can write down the variation equation of $\mathcal{S}_\Lambda(f)$ over $f'(x)$,

$$\int_{y\neq x} \mathrm{d}y(y-x)\left(\log\left|\frac{y-x}{\Lambda}\right| - 1\right) f''(y) = \sigma'\left(f'(x)\right). \tag{9.5}$$

By defining

$$[\mathbf{X}f](x) := \int_{y\neq x} \mathrm{d}y(y-x)\left(\log\left|\frac{y-x}{\Lambda}\right| - 1\right) f''(y), \tag{9.6}$$

we see that at the saddle point $f = f_\star$,

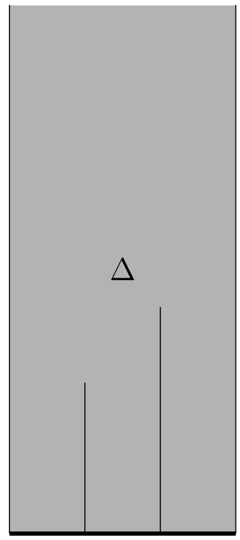
$$[Xf_\star](x) = \xi_i, \quad \text{for } -N + 2i - 2 < f'_\star(x) < -N + 2i, \tag{9.7}$$

$$\xi_i < [Xf_\star](x) < \xi_{i+1}, \quad \text{for } f'_\star(x) = -N + 2i, \tag{9.8}$$

where i runs from 0 to N and we set $\xi_0 = \xi_N = \infty$. We can then define

$$\varphi(x) := f'_\star(x) + \frac{1}{\pi i}[Xf_\star]'(x), \tag{9.9}$$

which is realized as a map from $\mathbb{R}$ to the boundary of the following region Δ,



(9.10)

as we can see that when $f'(x)$ does not belong to $\{-N+2i\}_{i=0}^{N}$, $\varphi(x)$ is real.

The next step is to construct a map to the above region Δ from the upper-half plane with cuts on the complex plane. Let us denote the conformal map as $\Phi(z)$, and it is constructed by the mathematicians to be

$$\Phi(z) = \frac{2}{\pi i} \log\left(w(z)\right) + N, \tag{9.11}$$

where $w(z)$ satisfies

$$\Lambda^N \left(w + \frac{1}{w}\right) = P_N(z), \tag{9.12}$$

for a polynomial $P_N(z)$ of the form

$$P_N(z)^2 - 4\Lambda^{2N} = \prod_{i=1}^{N} (z - \alpha_i^+)(z - \alpha_i^-). \tag{9.13}$$

We note that the map

$$g(w):\ w \mapsto \Lambda^N \left(w + \frac{1}{w}\right), \tag{9.14}$$

known as the Zhukowski function maps the open disk $|w| < 1$ to the exterior of the interval $[-2\Lambda^N, 2\Lambda^N]$, i.e. $\mathbb{C}\backslash[-2\Lambda^N, 2\Lambda^N]$.The Zhukowski function maps the real axis to the real axis, so the cut $[-2\Lambda^N, 2\Lambda^N]$ corresponds to $|w| \geq 1$ on the real axis. It is thus easy to see that on the z-plane, $\alpha_i^\pm$ are mapped to the endpoints of the corresponding N cuts. We pair α_i^- and α_i^+ together so that $P_N(z)^2 - 4\Lambda^{2N} < 0$ for $\alpha_i^- < z < \alpha_i^+$ (see Figure 2). When $P_N(z)^2 - 4\Lambda^{2N} < 0$, it implies that $|g(w)| = |P_N(z)| < 2\Lambda^N$, and it coincides with one cut.

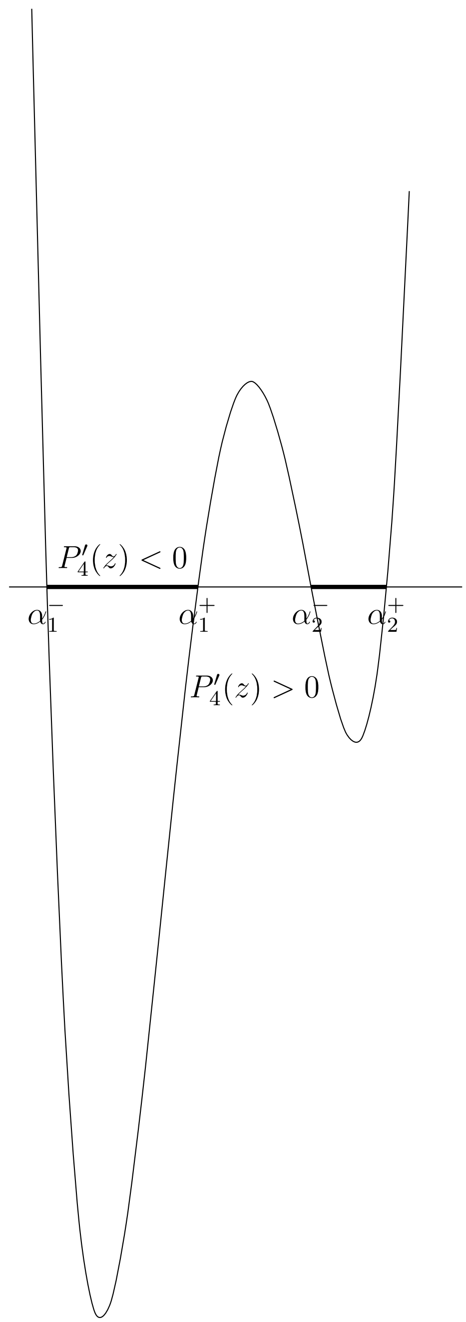


Figure 2: A typical Zhukowski function $g(w) = P_2(z)$ with $P_4'(z) := P_2(z)^2 - 4\Lambda^4 = (z - \alpha_1^-)(z - \alpha_1^+)(z - \alpha_2^-)(z - \alpha_2^+)$ on the real axis. The cut is marked black on the real axis.

Let us take for example $z = \alpha_i^- + \theta$, then at the first order of θ, we have

$$P_N(z)^2 - 4\Lambda^{2N} \simeq \theta(\alpha_i^- - \alpha_i^+) \prod_{j\neq i} (\alpha_i^- - \alpha_j^-)(\alpha_i^- - \alpha_j^+), \tag{9.15}$$

$$P_N(z) \simeq 2\Lambda^N + \frac{\theta}{2\Lambda^N}(\alpha_i^- - \alpha_i^+) \prod_{j\neq i} (\alpha_i^- - \alpha_j^-)(\alpha_i^- - \alpha_j^+), \tag{9.16}$$

and w can be solved to

$$w \simeq e^{2\pi i \ell} - \frac{\theta^{\frac{1}{2}}}{\sqrt{2}\Lambda^N} \left((\alpha_i^- - \alpha_i^+) \prod_{j\neq i} (\alpha_i^- - \alpha_j^-)(\alpha_i^- - \alpha_j^+) \right)^{\frac{1}{2}}, \tag{9.17}$$

for some integer ℓ. Note taht when $\theta > 0$, terms in $(\bullet)^{\frac{1}{2}}$ is negative, it is then easy to check that the cuts are mapped to the real axis in Δ.

Let us analyze the property of periods on the curve (9.12). The Coulomb branch parameters $\mathfrak{a}_i$ are fixed to coincide with the A-cycle

$$\mathfrak{a}_i = \oint_{A_i} \mathrm{d}S, \tag{9.18}$$

where

$$\mathrm{d}S = \frac{1}{2\pi i} \frac{z \mathrm{d}w}{w}. \tag{9.19}$$

Around the cuts in Δ, we actually have

$$\varphi(x) = \Phi(x + i0), \tag{9.20}$$

and also for real-valued ξ_i's, the maximizer satisfies

$$f'_\star(x) = \mathrm{Re}\varphi(x), \tag{9.21}$$

Further note that Φ' is real along the cuts (real axis), we have

$$\Phi'(z) = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{\mathrm{Re}\varphi'(x)}{(x - z)}, \tag{9.22}$$

and we can define a resolvent for any given function $f(x)$ as

$$Rf(z) := \frac{1}{2} \int_{\mathbb{R}} \frac{f''(x)}{z - x}, \tag{9.23}$$

to write down

$$\Phi'(z) = -\frac{2}{\pi i} Rf_\star(z) = \frac{2}{\pi i} \frac{w'(z)}{w(z)} = \frac{2}{\pi i} \frac{\mathrm{d}}{\mathrm{d}z} \log w(z). \tag{9.24}$$

The A-period can then be rewritten in the language of partition function as

$$a_i = \frac{1}{2\pi i}\oint_{A_i} zRf_\star(z)\mathrm{d}z. \tag{9.25}$$

As for the B-period, since we have

$$\mathrm{Im}\varphi(x) = -\frac{1}{\pi}[Xf_\star]'(x), \tag{9.26}$$

and $[Xf_\star](x) = \xi_i$ at $z = \alpha_i^- - 0$, we can write

$$\xi_{i+1} - \xi_i = -\pi\int_{\alpha_i^+}^{\alpha_{i+1}^-} \mathrm{Im}\varphi(x)\mathrm{d}x. \tag{9.27}$$

On the interval $(\alpha_i^+, \alpha_{i+1}^-)$, the function $\varphi(x)$ is pure imaginary, so integrating the above integral by parts, we obtain

$$\xi_{i+1} - \xi_i = -\pi i\int_{\alpha_i^+}^{\alpha_{i+1}^-} x\mathrm{d}\varphi = -4\pi i\int_{\alpha_i^+}^{\alpha_{i+1}^-} \mathrm{d}S = 2\pi i\oint_{B_i}\mathrm{d}S. \tag{9.28}$$

We remark that $\xi_{i+1} - \xi_i$ is the dual parameter to $\mathfrak{a}_{i+1} - \mathfrak{a}_i$ in the prepotential. This shows how the Seiberg-Witten theory is derived from the prepotential, and thus the partition function.

Adding matters Adding hypermultiplets in fundamental representation introduces a factor,

$$\prod_{f=1}^{N_f}[\phi_i - m_f], \tag{9.29}$$

into the instanton integrand Z_k, and the perturbative part is given by

$$Z^{pert}_{fund.}(\{m_f\}; R) = \exp\left(-\sum_{a=1}^{N}\sum_{f=1}^{N_f}\gamma_{\epsilon_1,\epsilon_2}(\mathfrak{a}_a - m_f)\right). \tag{9.30}$$

The free energy computed from the instanton partition function reads

$$\mathcal{E}_\Lambda = \int \mathrm{d}x\mathrm{d}y\frac{\rho(x)\rho(y)}{[x-y]^2} + \sum_{a=1}^{N}\int \mathrm{d}x\rho(x)\left(\log\left(\frac{[\mathfrak{a}_a - x][x-\mathfrak{a}_a]}{\Lambda^2}\right)\right) - \sum_{f=1}^{N_f}\int \mathrm{d}x\rho(x)\log\frac{[x-m_f]}{\Lambda}, \tag{9.31}$$

where the instanton counting parameter is reparameterized to $\Lambda^{2Nk-N_f k}$.

$$\begin{aligned}\mathcal{E}^{4d}_{\Lambda} = &-\int \mathrm{d}x\mathrm{d}y\rho''(x)\rho''(y)k_{\Lambda}(x-y) + \sum_{a=1}^{N}\int \mathrm{d}x\left(\rho''(x)k_{\Lambda}(x-\mathfrak{a}_a)+\rho''(x)k_{\Lambda}(\mathfrak{a}_a - x)\right)\\ &-\sum_{f=1}^{N_f}\int \mathrm{d}x\rho''(x)k_{\Lambda}(x-m_f)\\ =&-\int \mathrm{d}x\mathrm{d}y f''(x)f''(y)k_{\Lambda}(x-y) - \int \mathrm{d}x f''(x)k_{\Lambda}(x-m_f) + \sum_{a,b=1}^{N}k_{\Lambda}(\mathfrak{a}_a-\mathfrak{a}_b)\\ &-\sum_{a=1}^{N}\sum_{f=1}^{N_f}k_{\Lambda}(\mathfrak{a}_a - m_f)\end{aligned} \tag{9.32}$$

The prepotential at the end is obtained as

$$\mathcal{F} = -\int \mathrm{d}x\mathrm{d}y f''(x)f''(y)k_{\Lambda}(x-y) - \sum_{f=1}^{N_f}\int \mathrm{d}x f''(x)k_{\Lambda}(x-m_f). \tag{9.33}$$

Taking variation about $f(x)$, we see that the maximizer $f(x) = f_{m\star}(x)$ can be obtained from that of the pure gauge theory, $f_{\star}(x)$ with a shift,

$$f''_{m\star}(x) + \sum_{f=1}^{N_f}\delta(x-m_f) = f''_{\star}(x). \tag{9.34}$$

We can define a new variable $\tilde{w}$ associated to $f_{m\star}$. It can be worked out, for example, from

$$-\frac{1}{\pi i}\int_{\mathbb{R}}\frac{f''_{\star}}{z-x}\mathrm{d}x = \frac{2}{\pi i}\frac{\mathrm{d}}{\mathrm{d}z}\log w(z), \tag{9.35}$$

to obtain

$$\begin{aligned}\frac{2}{\pi i}\frac{\mathrm{d}}{\mathrm{d}z}\log\tilde{w}(z) = -\frac{1}{\pi i}\int_{\mathbb{R}}\frac{f''_{m\star}}{z-x}\mathrm{d}x &= -\frac{1}{\pi i}\int_{\mathbb{R}}\frac{f''_{\star}}{z-x}\mathrm{d}x + \sum_{f}\frac{1}{\pi i}\frac{1}{z-m_f}\\ &= \frac{2}{\pi i}\frac{\mathrm{d}}{\mathrm{d}z}\log w(z) + \sum_{f}\frac{1}{\pi i}\frac{\mathrm{d}}{\mathrm{d}z}\log(z-m_f),\end{aligned} \tag{9.36}$$

i.e.

$$\tilde{w}(z) = w(z)\prod_{f}(z-m_f)^{\frac{1}{2}}. \tag{9.37}$$

The Seiberg-Witten curve is then given by

$$\Lambda^{N-\frac{N_f}{2}}\left(\tilde{w} + \frac{\prod_{f=1}^{N_f}(z-m_f)}{\tilde{w}}\right) = \tilde{P}_N(z). \tag{9.38}$$

10 A Short Cut to Seiberg-Witten Curve

The above approach from the prepotential to the Seiberg-Witten curve is very systematic, rigorous but mathematical and seems to be artificial. In fact there is a easier and more clear way to carry out essentially the same analysis.

We introduce the resolvent

$$R(z) = \int_{-\infty}^{\infty} \frac{f''(x)}{z-x} \mathrm{d}x, \tag{10.1}$$

for the profile function $f(z)$, which is essentially the same as (9.23).

A well-known fact for the resolvent is that if $f''(x)$ has a finite support $[a_-, a_+]$, then $R(z)$ will have a branch cut between $[a_-, a_+]$. As we know that

$$f''(x) = \rho''(x) - \sum_a \delta(x - \mathfrak{a}_a), \tag{10.2}$$

and

$$\rho(x) = \epsilon_1\epsilon_2 \sum_{i=1}^{k} \delta(x - \phi_i), \tag{10.3}$$

before we take the thermodynamic limit, sitting on each pole of the contour integral,

$$\phi_{(a,x)} = \mathfrak{a}_a + \epsilon_1(i-1) + \epsilon_2(j-1), \tag{10.4}$$

where $x = (i,j) \in \lambda_a$ with $\sum_a |\lambda_a| = k$. Therefore in the computation of U(N) theory, the resolvent at the saddle point,

$$R_\star(z) = \int_{-\infty}^{\infty} \frac{f''_\star(x)}{z-x} \mathrm{d}x, \tag{10.5}$$

is expected to have N branch cuts, which are denoted as $\mathcal{C}_i = \left[a_-^i, a_+^i\right]$.

Also due to the fact that $f''(z)$ only has supports on $\mathcal{C}_i$'s, we obtain the following properties of $f''(x)$ by using the explicit form of $\rho(x)$,

$$\int_{\mathcal{C}_i} f''(x)\mathrm{d}x = -1, \tag{10.6}$$

$$\int_{\mathcal{C}_i} x f''(x)\mathrm{d}x = -\mathfrak{a}_i, \tag{10.7}$$

$$\int_{\mathcal{C}_i} x^2 f''(x)\mathrm{d}x = -\mathfrak{a}_i^2 + 2\epsilon_1\epsilon_2|\lambda_i|, \tag{10.8}$$

$$\begin{aligned}
\int_{\mathcal{C}_i} x^3 f''(x)\mathrm{d}x &= -\mathfrak{a}_i^3 + 6\epsilon_1\epsilon_2 \sum_{j=1}^{k_i} \phi_j \\
&= -\mathfrak{a}_i^3 + 6\epsilon_1\epsilon_2\mathfrak{a}_i|\lambda_i| + 3\epsilon_1\epsilon_2 \sum_n \epsilon_1 \left((\lambda_i^{(n)})^2 - \lambda_i^{(n)} \right) + 3\epsilon_1\epsilon_2 \sum_m \epsilon_2 \left((\lambda_i^{t(m)})^2 - \lambda_i^{t(m)} \right) \\
&= -\mathfrak{a}_i^3 + 6\epsilon_1\epsilon_2\mathfrak{a}_i|\lambda_i| - 3\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2)|\lambda_i| + 3\epsilon_1\epsilon_2 \left(\epsilon_1||\lambda_i||^2 + \epsilon_2||\lambda_i^t||^2 \right).
\end{aligned} \tag{10.9}$$

The branch cut of the resolvent can be seen in the following way: we split the resolvent into two parts, and call them the regular and singular parts of the resolvent. On $z \in \mathcal{C}_i$, we have

$$R_{reg}(z) := \lim_{\epsilon \to 0^+} \frac{1}{2} \left(R(z + i\epsilon) + R(z - i\epsilon) \right), \tag{10.10}$$

$$R_{sing}(z) := \lim_{\epsilon \to 0^+} \frac{1}{2i} \left(R(z + i\epsilon) - R(z - i\epsilon) \right). \tag{10.11}$$

If we simply take $R(z) = x^{\frac{1}{2}}$ or $R(z) = \log(z)$, then we can see that $R_{reg}(z)$ and $R_{sing}(z)$ are nothing but the real and imaginary part of $R(z)$. It is then natural to analytically continue these functions to the whole complex plane $z \in \mathbb{C}$.

We further define

$$R^{\pm}(z) := R_{reg}(z) \pm i R_{sing}(z). \tag{10.12}$$

By construction, $R_{reg}(z, \epsilon)$ is a smooth function even along the branch cuts $\mathcal{C}_i$'s, while R_{sing} changes its sign by crossing the branch cuts, i.e.

$$\lim_{\delta \to 0^+} R_{reg}(z + i\delta) = \lim_{\delta \to 0^+} R_{reg}(z - i\delta), \tag{10.13}$$

$$\lim_{\delta \to 0^+} R_{sing}(z + i\delta) = - \lim_{\delta \to 0^+} R_{sing}(z - i\delta). \tag{10.14}$$

For square-root branch cuts, $R^{\pm}(z)$ respectively give the single-valued function $R(z)$ defined on the two-sheet Riemann surface connected with branch cuts $\mathcal{C}_i$'s.

We need to analyze the asymptotic behavior of the resolvent, which is determined by the saddle point equation obtained by minimizing the prepotential with two Langrange multipliers,

$$-\frac{1}{\hbar^2} \int \mathrm{d}x\mathrm{d}y f''(y) k_\Lambda(x-y) + \sum_a \xi_a \left(\int_{\mathcal{C}_a} \mathrm{d}x \, x f''(x) + \mathfrak{a}_a \right) + \sum_a \eta_a \left(\int_{\mathcal{C}_a} \mathrm{d}x f''(x) + 1 \right). \tag{10.15}$$

The saddle point equation then reads

$$-\frac{2}{\hbar^2} \int \mathrm{d}y f''(y) k_\Lambda(x-y) + x\xi_i + \eta_i = 0, \tag{10.16}$$

for $x \in \mathcal{C}_i$. Taking derivative over x gives,

$$\int \mathrm{d}y f''(y) \left((x-y) \log\left(\frac{x-y}{\Lambda} \right) - (x-y) \right) = \frac{\hbar^2}{2} \xi, \tag{10.17}$$

$$\int \mathrm{d}y f''(y) \log\left(\frac{x-y}{\Lambda} \right) = 0, \tag{10.18}$$

$$\int \mathrm{d}y f''(y) \frac{1}{x-y} = 0. \tag{10.19}$$

We have to be careful that all the integrals above are principle-value integral. From the last equation, we conclude that for $z \in \mathbb{R} \setminus \cup_i \mathcal{C}_i$,

$$R_{reg\ \star}(z) = \lim_{\epsilon \to 0^+} \frac{1}{2} \left(R_\star(z + i\epsilon) + R_\star(z - i\epsilon) \right) = P.V. \int \mathrm{d}y f''_\star(y) \frac{1}{x-y} = 0$$
$$\Rightarrow R^+_\star(z) = -R^-_\star(z). \tag{10.20}$$

We also see that the asymptotic behavior of the resolvent function reads

$$\lim_{z\to\pm\infty} R_\star(z) = 0. \tag{10.21}$$

Let us define the A-cycles as contours surrounding the supports $\mathcal{C}_i$'s, and denote the one around $\mathcal{C}_i$ as $\mathcal{A}_i$. We then have

$$\oint_{\mathcal{A}_a} \mathrm{d}z R_\star(z) = \int_{-\infty}^{\infty} \mathrm{d}x \oint_{\mathcal{A}_a} \mathrm{d}z \frac{f_\star''(x)}{z-x} = 2\pi i \int_{\mathcal{C}_a} f_\star''(x)\mathrm{d}x = -2\pi i, \tag{10.22}$$

$$\oint_{\mathcal{A}_a} \mathrm{d}z\; z R_\star(z) = 2\pi i \int_{\mathcal{C}_a} x f_\star''(x)\mathrm{d}x = -2\pi i \mathfrak{a}_a. \tag{10.23}$$

The B-cycles $\mathcal{B}_i$ are defined as the circles that surround the endpoints of two neighboring branch cuts, α_+^i and α_-^{i+1}. We can evaluate the integral of resolvent in B-cycles as follows,

$$\begin{aligned}\oint_{\mathcal{B}_i} \mathrm{d}z R_\star(z) &= \int_{\alpha_i^+}^{\alpha_{i+1}^-} R_\star^+(z)\mathrm{d}z + \int_{\alpha_{i+1}^-}^{\alpha_i^+} R_\star^-(z)\mathrm{d}z \\ &= 2\int_{\alpha_i^+}^{\alpha_{i+1}^-} R_\star^+(z)\mathrm{d}z = 0,\end{aligned} \tag{10.24}$$

$$\begin{aligned}\oint_{\mathcal{B}_i} \mathrm{d}z\; z R_\star(z) &= 2\int_{\alpha_i^+}^{\alpha_{i+1}^-} z R_\star^+(z)\mathrm{d}z = 2\int_{-\infty}^{\infty}\mathrm{d}x \int_{\alpha_i^+}^{\alpha_{i+1}^-} \mathrm{d}z \frac{z f_\star''(x)}{z-x} \\ &= -2\int_{-\infty}^{\infty}\mathrm{d}x \int_{\alpha_i^+}^{\alpha_{i+1}^-} \mathrm{d}z f_\star''(x)\log(z-x) + 2\int_{-\infty}^{\infty}\mathrm{d}x \left[z\log(z-x)\right]_{\alpha_i^+}^{\alpha_{i+1}^-} f_\star''(x) \\ &= -2\int_{-\infty}^{\infty}\mathrm{d}x f_\star''(x)\left[(z-x)\log(z-x) - z\right]_{\alpha_i^+}^{\alpha_{i+1}^-} + 2\int_{-\infty}^{\infty}\mathrm{d}x\left[z\log(z-x)\right]_{\alpha_i^+}^{\alpha_{i+1}^-} f_\star''(x) \\ &= 2\int_{-\infty}^{\infty}\mathrm{d}x f_\star''(x)\left(x\log(\alpha_{i+1}^- - x) + \alpha_{i+1}^-\right) - 2\int_{-\infty}^{\infty}\mathrm{d}x f_\star''(x)\left(x\log(\alpha_i^+ - x) + \alpha_i^+\right).\end{aligned} \tag{10.25}$$

For example,

$$\begin{aligned}\int_{-\infty}^{\infty}\mathrm{d}x f_\star''(x)\left(x\log(\alpha_{i+1}^- - x) + \alpha_{i+1}^-\right) &= -\frac{\hbar^2}{2}\xi_{i+1} + \int_{-\infty}^{\infty}\mathrm{d}x f_\star''(x)\alpha_{i+1}^-\log(\alpha_{i+1}^- - x) \\ &\quad - \int_{-\infty}^{\infty}\mathrm{d}x f_\star''(x)(\alpha_{i+1}^- - x)(\log\Lambda + 1) + \int_{-\infty}^{\infty}\mathrm{d}x f_\star''(x)\alpha_{i+1}^- \\ &= -\frac{\hbar^2}{2}\xi_{i+1} + \int_{-\infty}^{\infty}\mathrm{d}x\; x f_\star''(x)(\log\Lambda + 1),\end{aligned} \tag{10.26}$$

where we used

$$\int_{-\infty}^{\infty}\mathrm{d}x f_\star''(x)\log(x-y) = \int_{-\infty}^{\infty}\mathrm{d}x f_\star''(x)\log(\Lambda). \tag{10.27}$$

We thus find that the B-cycle is given by

$$\oint_{\mathcal{B}_i}\mathrm{d}z\; z R_\star(z) = -\hbar^2(\xi_{i+1} - \xi_i). \tag{10.28}$$

This reproduces the well-known result of Seiberg-Witten curve. Now let us try to derive the curve it self from the resolvent. To do so, we need to construct some coordinate system on the curve. A candidate is

$$\int_0^z R_\star(z')\mathrm{d}z', \tag{10.29}$$

but depending on the integral contour, there is an ambiguity, i.e. by going around any A-cycle for n times, we see that

$$\int_0^z R_\star(z')\mathrm{d}z' \to \int_0^z R_\star(z')\mathrm{d}z' - 2\pi i n. \tag{10.30}$$

This logarithmic behavior can be suppressed by taking its exponential,

$$t_\pm(z) := \exp\left(-\int_0^z R_\star^\pm(z')\mathrm{d}z'\right). \tag{10.31}$$

It is then easy to see that the Seiberg-Witten one-form is given by

$$\lambda_{sw}^\pm = zR_\star^\pm(z)\mathrm{d}z = z\mathrm{d}\log(t_\pm(z)), \tag{10.32}$$

which is just the well-known expression. Once we could show that there are only square-root branch cuts in the resolvent $R(z)$, we can claim that

$$P(z) = t_+(z) + t_-(z), \tag{10.33}$$

is completely a well-defined and holomorphic function on $\mathbb{C}$, and then it has to be a Laurent polynomial of z. It is also easy to confirm that

$$t_+(z)t_-(z) = \exp\left(-\int_0^z (R_\star^+(z') + R_\star^-(z'))\mathrm{d}z'\right) = \exp\left(-2\int_0^z R_{reg\ \star}(z')\mathrm{d}z'\right) = 1. \tag{10.34}$$

$t_\pm(z)$ then satisify the following quadratic equation,

$$t^2 - P(z)t + 1 = 0, \tag{10.35}$$

or

$$t + \frac{1}{t} = P(z), \tag{10.36}$$

which is nothing but the Seiberg-Witten curve. The only remaining thing to do is to show that $P(z)$ is indeed a polynomial of z.

In the limit of $z \to \infty$, we have

$$R_\star(z) \sim \frac{1}{z}\int_{-\infty}^{\infty} f_\star''(x)\mathrm{d}x = -\frac{N}{z}, \tag{10.37}$$

and thus

$$t_\pm(z) \to z^N, \quad z \to \pm\infty, \tag{10.38}$$

which indicates that $P(z)$ is a polynomial of z of degree N (as we can further show that there is no singularity in $P(z)$).

Adding matter As discussed before, adding matter deforms the prepotential to

$$\mathcal{F} = -\int \mathrm{d}x\mathrm{d}y f''(x)f''(y)k_\Lambda(x-y) + \sum_{f=1}^{N_f}\int \mathrm{d}x f''(x)k_\Lambda(x-m_f). \tag{10.39}$$

The saddle-point equation is now modified to

$$-\frac{2}{\hbar^2}\int \mathrm{d}y f''(y)k_\Lambda(x-y) + \frac{1}{\hbar^2}\sum_{f=1}^{N_f} k_\Lambda(x-m_f) + x\xi_i + \eta_i = 0, \tag{10.40}$$

and

$$\int \mathrm{d}y \left(f''(y) - \frac{1}{2}\sum_f \delta(y-m_f)\right)\left((x-y)\log\left(\frac{x-y}{\Lambda}\right) - (x-y)\right) = \frac{\hbar^2}{2}\xi_i, \tag{10.41}$$

$$\int \mathrm{d}y \left(f''(y) - \frac{1}{2}\sum_f \delta(y-m_f)\right)\log\left(\frac{x-y}{\Lambda}\right) = 0, \tag{10.42}$$

$$\int \mathrm{d}y \left(f''(y) - \frac{1}{2}\sum_f \delta(y-m_f)\right)\frac{1}{x-y} = 0, \tag{10.43}$$

for $x \in \mathcal{C}_i$.

Since adding matter does not change the pole structure of the instanton partition function, the A-cycle and B-cycle integral of the resolvent does not change. However, one can further perform a contour integral around $z = m_f$, (?)

$$\oint_{m_f} \mathrm{d}z R_\star(z) = \int_{-\infty}^{\infty}\mathrm{d}x \oint_{m_f}\mathrm{d}z \frac{f''_\star(x)}{z-x} = f''_\star(m_f). \tag{10.44}$$

The regular part of the resolvent now satisfies

$$R_{reg\ \star}(z) = \lim_{\epsilon\to 0^+}\frac{1}{2}\left(R_\star(z+i\epsilon) + R_\star(z-i\epsilon)\right) = P.V.\int \mathrm{d}y f''_\star(y)\frac{1}{z-y} = -\frac{1}{2}\sum_f \frac{1}{z-m_f}, \tag{10.45}$$

and thus

$$t_+(z)t_-(z) = \exp\left(-2\int_0^z R_{reg\ \star}(z')\mathrm{d}z'\right) = \prod_f (1 - z/m_f). \tag{10.46}$$

The Seiberg-Witten curve is then modified to

$$t^2 - P(z)t + \prod_f (1-z/m_f) = 0. \tag{10.47}$$

uplift to 5d The computation in 5d is almost parallel. In 4d, the kernel function, $k_\Lambda(x) := \frac{x^2}{2}\left(\log\frac{|x|}{\Lambda} - \frac{3}{2}\right)$, was determined from

$$k_\Lambda^{(4)} = -\frac{1}{x^2}, \tag{10.48}$$

and in 5d, we need to solve

$$\mathcal{K}_\Lambda^{(4)}(x) = -\frac{1}{[x]^2}. \tag{10.49}$$

A candidate for $\mathcal{K}_\Lambda$ (up to some integral constants) is

$$\mathcal{K}(x) = -Li_3(e^{-x}). \tag{10.50}$$

(How to relate to the 4d kernel function?)

The resolvent function is now defined as

$$R(z) = \int_{-\infty}^{\infty} \frac{f''(x)}{1-e^{-(z-x)}}\mathrm{d}x. \tag{10.51}$$

The saddle point equation is almost the same,

$$-\frac{2}{\hbar^2}\int \mathrm{d}y f''(y)\mathcal{K}_\Lambda(x-y) + x\xi_i + \eta_i = 0. \tag{10.52}$$

As we have seen in the 4d case that the constant Λ does not affect the final result (one may absorb it into the integral variable x), we simply use $\mathcal{K}(x)$ to perform the computation. Then

$$\int \mathrm{d}y f''(y) Li_2(e^{-(x-y)}) = \frac{\hbar^2}{2}\xi, \tag{10.53}$$

$$\int \mathrm{d}y f''(y)\log\left(1-e^{-(x-y)}\right) = 0, \tag{10.54}$$

$$\int \mathrm{d}y f''(y)\frac{e^{-(x-y)}}{1-e^{-(x-y)}} = 0. \tag{10.55}$$

The calculations involving A-cycles remain to be the same,

$$\oint_{\mathcal{A}_a} \mathrm{d}z R_\star(z) = \int_{-\infty}^{\infty}\mathrm{d}x \oint_{\mathcal{A}_a}\mathrm{d}z \frac{f_\star''(x)}{1-e^{-(z-x)}} = 2\pi i\int_{\mathcal{C}_a} f_\star''(x)\mathrm{d}x = -2\pi i, \tag{10.56}$$

$$\oint_{\mathcal{A}_a}\mathrm{d}z\ zR_\star(z) = 2\pi i\int_{\mathcal{C}_a} x f_\star''(x)\mathrm{d}x = -2\pi i\mathfrak{a}_a, \tag{10.57}$$

as

$$\oint \mathrm{d}x\frac{1}{1-e^{-x}} = \oint \mathrm{d}e^x\frac{1}{e^x-1} = 1. \tag{10.58}$$

The B-cycle integral also gives the same result.

The asymptotic behavior with matter is modified to

$$R_{reg\ \star}(z)=\lim_{\epsilon\to 0^+}\frac{1}{2}\left(R_\star(z+i\epsilon)+R_\star(z-i\epsilon)\right)=P.V.\int \mathrm{d}y f''_\star(y)\frac{1}{1-e^{-(z-y)}}=-\frac{1}{2}\sum_f\frac{1}{1-e^{-(z-m_f)}}. \tag{10.59}$$

An interesting phenomenon occurs that $R_{reg\ \star}(z)$ behaves differently in the limit $m_f\to\pm\infty$. Since we have

$$\int \mathrm{d}z\frac{1}{1-e^{-(z-m)}}=\log(e^m-e^z), \tag{10.60}$$

then

$$t_+(z)t_-(z)=\exp\left(-2\int_0^z R_{reg\ \star}(z')\mathrm{d}z'\right)=\prod_f\frac{1-e^{z-m_f}}{1-e^{-m_f}}. \tag{10.61}$$

The 5d SW curve

$$t^2-P_{5d}(z)t+\prod_f\frac{1-e^{z-m_f}}{1-e^{-m_f}}=0. \tag{10.62}$$

When we take $m_f\to\infty$, the corresponding hypermultiplet decouples, but when $m_f\to-\infty$, an additional factor e^z is introduced into the equation. As the contribution of the Chern-Simons level to the curve is also given by the form $e^{\kappa z}$ at the same position, one may interpret the movement of the mass of one hypermultiplet from ∞ to $-\infty$ as changing the Chern-Simons level by one.

10.1 Derivation from the explicit expression of partition function

When we perform the contour integral in the partition function explicitly, we obtain an expression as a combination of the following factor,

$$N_{\lambda\mu}(Q):=\prod_{i,j=1}^{\infty}\left(1-Qq^{i+j-\lambda_i-\mu_j-1}\right). \tag{10.63}$$

This factor is equivalent to the so-called Nekrasov factor up to the perturbative contribution $N_{\emptyset\emptyset}(Q)$ (and transpose of Young diagram). The key identity to analyze the thermodynamic behavior of the partition function is

$$N_{\lambda\mu}(e^{-r(a-b)})=\exp\left[\frac{1}{4}\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}y f''_\lambda(x-a|\hbar)f''_\mu(y-b|\hbar)\gamma_\hbar(x-y)\right], \tag{10.64}$$

with the explicit form of $\gamma_\hbar$ (in 5d) known as

$$\gamma_\hbar(x):=\sum_{n=1}^{\infty}\frac{1}{n}\frac{e^{-rnx}}{(1-e^{-rn\hbar})(1-e^{rn\hbar})}. \tag{10.65}$$

11 BCD-type Seiberg-Witten curves

11.1 Example: SO(4) theory

The prepotential of SO($2N+\chi$) theory is easily computed to

$$\mathcal{F} = -\frac{1}{8}\int \mathrm{d}x\mathrm{d}y f''_{SO}(x) f''_{SO}(y) k_\Lambda(x-y) - \left(1-\frac{\chi}{2}\right)\int \mathrm{d}x f''_{SO}(x) k_\Lambda(x), \tag{11.1}$$

where we shall define

$$\rho_{SO}(x) = \epsilon_1\epsilon_2 \sum_{i=1}^{k} \left(\delta(x-\phi_i) + \delta(x+\phi_i)\right), \tag{11.2}$$

and

$$f_{SO}(x) = -2\rho_{SO}(x) + \sum_{a=1}^{N} \left(|x-\mathfrak{a}_a| + |x+\mathfrak{a}_a|\right), \tag{11.3}$$

satisfying $f_{SO}(x) = f_{SO}(-x)$ and $\rho_{SO}(x) = \rho_{SO}(-x)$.

The prepotential can be identified as that of SU($2N$) with Coulomb branch parameters $\{\pm\mathfrak{a}_a\}$ and with $4-2\chi$ massless hypermultiplets. Therefore we can write the algebraic curve as

$$\Lambda^{2N+\chi-2}\left(w + \frac{z^{4-2\chi}}{w}\right) = P_{2N}(z), \tag{11.4}$$

where $P_{2N}(z)$ is symmetric about $z \leftrightarrow -z$, and thus is a polynomial of z^2.

For SO(4) theory, the curve is given by

$$\Lambda^2\left(w + \frac{z^4}{w}\right) = z^4 + u_1 z^2 + u_2. \tag{11.5}$$

In the case of $N=2$ and $\chi=0$, i.e. SO(4) theory, it can be alternatively realized by two independent SU(2) theories, SO(4)=SU(2)×SU(2).

$$\mathcal{F}_{SO(4)} = \mathcal{F}_{SU(2)_1} + \mathcal{F}_{SU(2)_2}. \tag{11.6}$$

11.2 Example: SO(6) theory

In fact, one can redefine

$$\tilde{f}''_{SO}(x) = f''_{SO}(x) + 2\delta(x), \tag{11.7}$$

to rewrite

$$\mathcal{F} = -\frac{1}{8}\int \mathrm{d}x\mathrm{d}y f''_{SO}(x) f''_{SO}(y) k_\Lambda(x-y) - \int \mathrm{d}x f''_{SO}(x) k_\Lambda(x)$$
$$= -\frac{1}{8}\int \mathrm{d}x\mathrm{d}y \tilde{f}''_{SO}(x)\tilde{f}''_{SO}(y) k_\Lambda(x-y), \tag{11.8}$$

where we used the even nature of $f_{SO}(x)$ and $k_\Lambda(0)=0$. It looks very similar to the prepotential of SU-type.

$$\tilde{f}_{SO}(x) = -2\rho_{SO}(x) + (|x\pm \mathfrak{a}_1| + |x\pm \mathfrak{a}_2| + |x\pm \mathfrak{a}_3|) + |x|. \tag{11.9}$$

11.3 Example: Sp(1) theory

In the case of Sp(N) theory, we instead define

$$f_{Sp}(x) = -2\rho_{SO}(x) + \sum_{a=1}^{N} (|x-\mathfrak{a}_a| + |x+\mathfrak{a}_a|) + 2|x|, \tag{11.10}$$

and then the prepotential is given by

$$\mathcal{F} = -\frac{1}{8}\int \mathrm{d}x\mathrm{d}y f''_{Sp}(x) f''_{Sp}(y) k_\Lambda(x-y). \tag{11.11}$$

We see that it is equivalent to a U($2N+2$) theory with Coulomb branch parameters specified to $\{\pm\mathfrak{a}_1, \pm\mathfrak{a}_2, \dots, \pm\mathfrak{a}_N, 0, 0\}$, however it is not completely correct to simply substitute the above values into the U($2N+2$) Seiberg-Witten curve.

Note that the branch cuts will be symmetric about $z\to -z$ on the z-plane, and we first consider a map $x=\varphi(z)$ with

$$\varphi:\ z\mapsto \frac{z}{\Lambda^2} + \beta^-. \tag{11.12}$$

The branch cuts are folded to the real axis, belonging to the interval $[\beta^-, \infty)$, and the pure imaginary axis is mapped to $(-\infty, \beta^-]$. To go to a region similar to Δ, we consider the following map,

$$\tilde{F} = \frac{1}{2\pi}\arccos\left(\frac{x\prod_{l=1}^{N}(x-\beta_l)}{2\Lambda^{N+1}}\right). \tag{11.13}$$

Note that

$$\cosh(x) = \cos(ix), \tag{11.14}$$

and we can analytically continue arccos to the whole complex plane by solving

$$2a = e^{ix} + e^{-ix}, \tag{11.15}$$

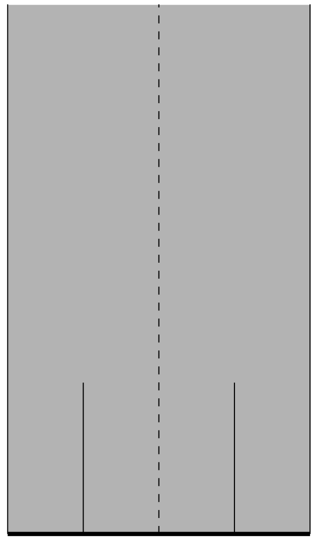


Figure 3: The cuts in Sp(N) theory (in this case is $N = 1$) after Zhukowski's map. The pure imaginary axis on the z-plane is mapped to the dashed line above.

to have

$$x = \frac{1}{i} \log \left(a \pm i\sqrt{1-a^2} \right). \tag{11.16}$$

We see that when $|a| < 1$,

$$a \pm i\sqrt{1-a^2}, \tag{11.17}$$

is on the unit circle, and thus x is a pure real number. While for $a < -1$,

$$a \pm \sqrt{a^2-1} < 0, \tag{11.18}$$

and $\mathrm{Re}(x) = \pi$ is fixed. Combining the map φ and $\tilde{F}$, we can map the z-plane to a ϑ-plane, where the cuts are all mapped to the intervals $\left[\frac{n}{2}, \frac{n+1}{2}\right]$ on the real axis (see Figure 3).

With the Zhukowski map,

$$\vartheta = F(z) = \tilde{F} \circ \varphi(z) =: \frac{1}{2\pi} \arccos \left(\frac{P_{Sp}(z)}{2\Lambda^{2N+2}} \right), \tag{11.19}$$

and defining

$$w := e^{2\pi i \vartheta}, \tag{11.20}$$

we obtained the Seiberg-Witten curve in the Sp-case,

$$\Lambda^{N+1} \left(w + \frac{1}{w} \right) = P_{Sp}(z), \tag{11.21}$$

where

$$P_{Sp}(z) = (z^2+\Lambda\beta^-)\prod_{l=1}^{N}(z^2+\Lambda\beta^- -\beta_l). \tag{11.22}$$

We further note that the endpoints of the branch cut interval, in particular for the first cut $\beta^\pm$, should satisfy

$$\beta^\pm\prod_{l=1}^{N}(\beta^\pm-\beta_l) = \pm(-1)^N 2\Lambda^{N+1}, \tag{11.23}$$

and therefore,

$$P_{Sp}(z) = z^2\prod_{l=1}^{N}(z^2-\alpha_l^2)+2(-1)^{N+1}\Lambda^{2N+2}, \tag{11.24}$$

for some parameters α_l's.

11.4 Sp(N) theories in 5d

There are four integrals for the instanton partition function of Sp(N) gauge theories in 5d. Depending on the number of instantons $k=2\ell+\chi$ for $\ell\in\mathbb{N}$ and $\chi=0,1$, the $O^+(k)$ piece of the partition function is given by the integrand,

$$\begin{aligned}Z^+_{vec} =&\frac{[2\epsilon_+]^\ell}{([\epsilon_1][\epsilon_2])^{\ell+\chi}}\left(\prod_{a=1}^{N}\frac{1}{[\pm\mathfrak{a}_a+\epsilon_+]}\prod_{I=1}^{\ell}\frac{[\pm\phi_I][\pm\phi_I+2\epsilon_+]}{[\pm\phi_I+\epsilon_1][\pm\phi_I+\epsilon_2]}\right)^\chi\\ &\times\prod_{I=1}^{\ell}\frac{1}{[\pm2\phi_I+\epsilon_1][\pm2\phi_I+\epsilon_2]\prod_{a=1}^{N}[\pm\phi_I\pm\mathfrak{a}_a+\epsilon_+]}\\ &\times\prod_{I<J}\frac{[\pm\phi_I\pm\phi_J][\pm\phi_I\pm\phi_J+2\epsilon_+]}{[\pm\phi_I\pm\phi_J+\epsilon_1][\pm\phi_I\pm\phi_J+\epsilon_2]},\end{aligned} \tag{11.25}$$

while for the $O^-(k)$ piece, when $k=2\ell+1$,

$$\begin{aligned}Z^-_{vec} =&\frac{[2\epsilon_+]^\ell}{([\epsilon_1][\epsilon_2])^{\ell+1}}\prod_{I=1}^{\ell}\frac{\mathrm{ch}(\pm\phi_I)\mathrm{ch}(\pm\phi_I+2\epsilon_+)}{\mathrm{ch}(\pm\phi_I+\epsilon_1)\mathrm{ch}(\pm\phi_I+\epsilon_2)\prod_{a=1}^{N}[\pm\phi_I\pm\mathfrak{a}_a+\epsilon_+]}\prod_{a=1}^{N}\frac{1}{\mathrm{ch}(\pm\mathfrak{a}_a+\epsilon_+)}\\ &\times\prod_{I=1}^{\ell}\frac{1}{[\pm2\phi_I+\epsilon_1][\pm2\phi_I+\epsilon_2]}\prod_{I<J}\frac{[\pm\phi_I\pm\phi_J][\pm\phi_I\pm\phi_J+\epsilon_+]}{[\pm\phi_I\pm\phi_J+\epsilon_1][\pm\phi_I\pm\phi_J+\epsilon_2]},\end{aligned} \tag{11.26}$$

and when $k=2\ell$,

$$\begin{aligned}Z^-_{vec} =&\frac{[2\epsilon_+]^{\ell-1}\mathrm{ch}(2\epsilon_+)}{[\epsilon_1]^\ell[\epsilon_2]^\ell[2\epsilon_1][2\epsilon_2]\prod_{a=1}^{N}[\pm2\mathfrak{a}_a+2\epsilon_+]}\prod_{I=1}^{\ell-1}\frac{[\pm2\phi_I][\pm2\phi_I+4\epsilon_+]}{[\pm2\phi_I+2\epsilon_1][\pm2\phi_I+2\epsilon_2]\prod_{a=1}^{N}[\pm\phi_I\pm\mathfrak{a}_a+\epsilon_+]}\\ &\times\prod_{I=1}^{\ell-1}\frac{1}{[\pm2\phi_I+\epsilon_1][\pm2\phi_I+\epsilon_2]}\prod_{I<J}\frac{[\pm\phi_I\pm\phi_J][\pm\phi_I\pm\phi_J+2\epsilon_+]}{[\pm\phi_I\pm\phi_J+2\epsilon_1][\pm\phi_I\pm\phi_J+2\epsilon_2]}.\end{aligned} \tag{11.27}$$

Taking the classical limit, we obtain

$$\mathcal{E}^{+} = \int \mathrm{d}x\mathrm{d}y \frac{\rho_{SO}(x)\rho_{SO}(y)}{[x-y]^2} + \int \mathrm{d}x 2\rho_{SO}(x) \log\frac{[2x]}{\Lambda} + \sum_{a=1}^{N} \int \mathrm{d}x \rho_{SO}(x) \log\frac{[x \pm \mathfrak{a}_a]}{\Lambda^2} + \epsilon_1\epsilon_2\chi \int \mathrm{d}x \frac{\rho_{SO}(x)}{[x]^2}. \quad (11.28)$$

$$\mathcal{E}^{-}_{odd} = \int \mathrm{d}x\mathrm{d}y \frac{\rho_{SO}(x)\rho_{SO}(y)}{[x-y]^2} + \int \mathrm{d}x 2\rho_{SO}(x) \log\frac{[2x]}{\Lambda} + \sum_{a=1}^{N} \int \mathrm{d}x \rho_{SO}(x) \log\frac{[x \pm \mathfrak{a}_a]}{\Lambda^2} + \epsilon_1\epsilon_2 \int \mathrm{d}x \frac{\rho_{SO}(x)}{\mathrm{ch}(x)^2}, \quad (11.29)$$

and

$$\mathcal{E}^{-}_{even} = \int \mathrm{d}x\mathrm{d}y \frac{\rho_{SO}(x)\rho_{SO}(y)}{[x-y]^2} + \int \mathrm{d}x 2\rho_{SO}(x) \log\frac{[2x]}{\Lambda} + \sum_{a=1}^{N} \int \mathrm{d}x \rho_{SO}(x) \log\frac{[x \pm \mathfrak{a}_a]}{\Lambda^2} + 4\epsilon_1\epsilon_2 \int \mathrm{d}x \frac{\rho_{SO}(x)}{[x]^2}. \quad (11.30)$$

The instanton counting parameter is scaled as $\mathfrak{q} = \Lambda^{4N+4}$. It seems (?) that only $\mathcal{E}^{-}_{even}$ contributes in the classical limit.

Before we analyze the saddle-point equation, let us first work on the contribution from the perturbative part. Recall that the perturbative contribution is governed by the function,

$$\gamma_0(x) = \frac{x^3}{12} - \frac{x^2}{2}\log(R\Lambda) - \frac{1}{R^2}\mathrm{Li}_3(e^{-Rx}), \quad (11.31)$$

and we note that it satisfies

$$\frac{\mathrm{d}^4}{\mathrm{d}x^4}\gamma_0(x) = -\frac{R^2}{4\sinh^2(Rx/2)} = -\frac{R^2}{[Rx]}, \quad (11.32)$$

we can identify $\gamma_0(x)$ as $\mathcal{K}_\Lambda(Rx)$ (note that before we worked at $R=1$).

Since the perturbative part in general is given by

$$Z^{pert}(\{\mathfrak{a}_a\}; R) \propto \exp\left(\sum_{\alpha:\ \mathrm{root}} \gamma_{\epsilon_1,\epsilon_2}(\mathfrak{a}_\alpha | R, \Lambda)\right), \quad (11.33)$$

indeed we can use the profile function

$$f_{Sp}(x) = -\rho_{SO}(x) + \sum_{a=1}^{N}\left(|x-\mathfrak{a}_a| + |x+\mathfrak{a}_a|\right) + 2|x|, \quad (11.34)$$

instead of ρ_{SO} to describe the prepotential simply as

$$\mathcal{F}_{Sp} = \int \mathrm{d}x\mathrm{d}y f''_{Sp}(x) f''_{Sp}(y) \mathcal{K}_\Lambda(x-y), \tag{11.35}$$

and the perturbative contribution just gives the remaining terms needed for completing the square.

The saddle-point equation is given by

$$-\frac{2}{\hbar^2}\int \mathrm{d}x\mathrm{d}y f''_{Sp}(y)\mathcal{K}_\Lambda(x-y) + \sum_a \xi_a \left(\int_{\mathcal{C}_{\pm a}} \mathrm{d}x\ x f''_{Sp}(x) \pm \mathfrak{a}_a\right) + \sum_a \eta_a \left(\int_{\mathcal{C}_{\pm a}} \mathrm{d}x f''_{Sp}(x) + 2\right)$$
$$+(\text{contributions from origin}) = 0, \tag{11.36}$$

where one may set $\mathfrak{a}_0 = 0$ to include the additional contributions from four additional

11.5 quiver gauge theory

Now we consider quiver gauge theories with gauge nodes V_i's connected by bifundamental hypermultiplets, which can be graphically represented by edges $e_{i\to j}$. The prepotential is thus given by

$$\mathcal{F} = -\sum_{V_i}\int \mathrm{d}x\mathrm{d}y f''_i(x) f''_i(y)\mathcal{K}_\Lambda(x-y) + \sum_{V_i; f_i=1}^{N_f^{(i)}} \int \mathrm{d}x f''_i(x)\mathcal{K}_\Lambda(x - m_f^{(i)})$$
$$+\sum_{e_{i\to j}} \int \mathrm{d}x\mathrm{d}y f''_i(x) f''_j(y) \mathcal{K}_\Lambda(x-y+m_e). \tag{11.37}$$

The saddle-point equation with constraints is thus given by

$$-\frac{2}{\hbar^2}\int \mathrm{d}y f''_i(y)\mathcal{K}_\Lambda(x-y) + \frac{1}{\hbar^2}\sum_{f=1}^{N_f^{(i)}} \mathcal{K}_\Lambda(x-m_f^{(i)}) + x\xi_a^{(i)} + \eta_a^{(i)}$$
$$+\sum_{e_{i\to j}}\int \mathrm{d}y f''_j(y)\mathcal{K}_\Lambda(x-y+m_e) + \sum_{e_{j\to i}}\int \mathrm{d}y f''_j(y)\mathcal{K}_\Lambda(y-x+m_e) = 0. \tag{11.38}$$

Taking three-times of derivatives over x, we obtain

$$\int \mathrm{d}y \left(f''_i(y) - \frac{1}{2}\sum_f \delta(y-m_f^{(i)})\right)\frac{e^{-(x-y)}}{1-e^{-(x-y)}} - \sum_{e_{i\to j}}\int \mathrm{d}y f''_j(y)\frac{e^{-(x-y+m_e)}}{1-e^{-(x-y+m_e)}}$$
$$-\sum_{e_{j\to i}}\int \mathrm{d}y f''_j(y)\frac{e^{-(y-x+m_e)}}{1-e^{-(y-x+m_e)}} = 0. \tag{11.39}$$

From this, we obtain

$$t_+^{(i)}(z)t_-^{(i)}(z) = P_{matter}^{(i)}(z)\prod_{e_{i\to j}} t_{reg}^{(j)}(z+m_e)\prod_{e_{j\to i}} t_{reg}^{(j)}(z-m_e). \tag{11.40}$$

12 Gaiotto form of the curve and Alday-Gaiotto-Tachikawa duality

Let us start with the Seiberg-Witten curve for SU(2) $N_f = 4$ described in previous section,

$$c_1 \frac{(x-\mu_1)(x-\mu_2)}{z} + c_2(x-\mu_3)(x-\mu_4)z = x^2 - u. \tag{12.1}$$

When all matter is massless, by appropriately rescaling z and u, we reach the following form of the curve introduced by Witten in arXiv:hep-th/9703166.

$$(z-1)(z-z_1)x^2 = uz, \tag{12.2}$$

together with the Seiberg-Witten differential $\lambda = x\frac{\mathrm{d}z}{z}$.

The Gaiotto form is obtained by further rescale $x \to (1/z)x$ and the curve now takes the form

$$x^2 = \frac{u}{z(z-1)(z-z_1)}, \tag{12.3}$$

with Seiberg-Witten differential $\lambda = x\mathrm{d}z$. There are four singualrity, which will be identified to the punctures in class $\mathcal{S}$ theories, at 0, 1, z_1 and ∞. We can make use of an $\mathrm{SL}(2,\mathbb{Z})$ transformation, $z = \frac{az'+b}{cz'+d}$ and $x = (cz+d)^2x'$, to put these poles in a more symmetrical way

$$x^2 = \frac{\tilde{u}}{\prod_{i=1}^4 (z-z_i)}, \tag{12.4}$$

where in terms of $\mathrm{SL}(2,\mathbb{Z})$, $z_1 = -b/a$, $z_2 = (d-b)/(a-c)$, $z_3 = (dz_1 - b)/(a - cz_1)$, $z_4 = -d/c$ and $\tilde{u} = u/(ac(a-c)(a-cz_1))$, without changing the Seiberg-Witten differential. This system has an $\mathrm{SL}(2,\mathbb{Z})$ symmetry and let us see how it acts on physical quantities of the gauge theory. We go back to the more traditional form of the SW curve (12.3), and to exchange the singularities at 1 and 0, we transform $(z,x) \to (1-z,x)$, under this transformation, z_1 goes to $1-z_1$ and $u \to -u$. Since it interchanges the branch point 0 and 1, we expect the contour of a and a_D also interchanges. With the transformation $\lambda \to -\lambda$, we have $a \to -a_D$ and $a_D \to a$[3], together with $\tau = \frac{a_D}{a} \to -\frac{1}{\tau}$. Another simple transformation we can have is $(z,x) \to (1/z, -z^2x)$. This interchanges 0 and ∞, and put z_1 into $1/z_1$. Considering the contour in details, we find another independent $\mathrm{SL}(2,\mathbb{Z})$ transformation, $\tau \to \frac{\tau}{\tau+1}$. We see that these $\mathrm{SL}(2,\mathbb{Z})$ dualities acting on z-coordinates, in fact generates the $\mathrm{SL}(2,\mathbb{Z})$ transformation on τ, i.e. the S-dualities.

More generally, Witten in arXiv:hep-th/9703166 gave the expression of SW-curves for quiver gauge theories with gauge group SU(2).

$$\prod_{a=0}^{n}(z-z_a)x^2 = U_{n-1}(z)z, \tag{12.5}$$

[3]We will see the minus sign coming out if we carefully trace how the contour is described in the transformed z-plane.

where the quiver gauge theory has n SU(2) gauge fields and $U_{n-1}(z)$ is a polynomial of z with degree $n-1$.

Let us go to the massive case. The curve now reads

$$(z-1)(z-z_1)x^2 - ((\mu_4+\mu_3)z^2 + (\mu_1+\mu_2)z_1)x + (\mu_3\mu_4 z^2 + \mu_1\mu_2 z_1) = uz. \tag{12.6}$$

Let us check the transformation $z \to 1/z$ for the curve. We clearly see that accompanied with the transformation $(\mu_1, \mu_2) \leftrightarrow (\mu_3, \mu_4)$ and $z_1 \to 1/z_1$, without changing x. This duality is inherited even in the less-matter case, when we put several mass parameters into infinity. For example in the pure gauge theory, we have

$$\Lambda^2(z + 1/z) = x^2 - u, \tag{12.7}$$

and it is clearly invariant under $z \leftrightarrow 1/z$. It is not clear how other $\mathrm{SL}(2,\mathbb{Z})$ transformations should be interpreted in the less-matter case.

Let us go back to the massive $N_f = 4$ case. The general case is

$$x = \frac{1}{z(z-1)(z-z_1)}\left((\mu_4+\mu_3)z^2 + (\mu_1+\mu_2)z \pm \sqrt{(\mu_3-\mu_4)^2 z^4 + \dots}\right), \tag{12.8}$$

where the convention for the Seiberg-Witten differential is $\lambda = x\mathrm{d}z$. In the generic case, i.e. not all masses are zero, we have four poles at $z = 0, 1, z_1, \infty$. For example, taking the + branch, we can evaluate the integral $\oint_{pole} \lambda$ as

$$2z_1\sqrt{\mu_1\mu_4}, \quad (\mu_1+\mu_2+\mu_3+\mu_4), \quad (\mu_4+\mu_3)z_1^2 + (\mu_1+\mu_2)z_1, \quad \mu_3+\mu_4+|\mu_3-\mu_4|. \tag{12.9}$$

Let us denote these residues as $m_{1,\dots,4}$.

...

It was noticed that since the 4d $\mathcal{N}=2$ SU(2) $N_f = 4$ SCFT can be constructed from the 6d $(2,0)$ SCFT by compactifying on the punctured sphere with 4 full punctures, and at the same time its partition function can be expressed as the four point function of a CFT on the sphere. Then it is natural to find a justification for this phenomenon.

Note that in the SU(2) $N_f = 4$ case, we have double poles around singularities in $\phi_2 := x^2$ (in the convention of $\lambda = x\mathrm{d}z$. We would like to identify them as the action of the 2d stress tensor around the punctures on the sphere. Then we see that

$$x^2\big|_{pole} = \langle T(z)\rangle\big|_{puncture} \sim \frac{m_i^2}{(z-z_i)^2} + \dots, \tag{12.10}$$

This is the typical behavior of a highest weight state with conformal dimension $h_i = m_i^2$ sitting at the puncture. From this observation, Gaiotto conculded that it was truly four vertex operators inserted in the sphere the situation that characterizes the 4d gauge theory, SU(2) $N_f = 4$.

In analogy with this prescription, we can consider the pure SU(2) gauge theory, whose SW curve in the same convention reads,

$$\phi_2 = \frac{\Lambda^2}{z^3} + \frac{u}{z^2} + \frac{\Lambda^2}{z}. \tag{12.11}$$

This quantity should be mapped to the VEV of $T(z) = \sum_n L_n z^{-n-2}$ and the most natural scenario is to have a vector $|G\rangle$ satisfying

$$L_1 |G\rangle = \Lambda^2 |G\rangle . \tag{12.12}$$

The vector $|G\rangle$ is usually called a Gaiotto state and a lot of studies showed that the inner product of two Gaiotto states agrees exactly with the Nekrasov's instanton partition function.

This can be explained in the original AGT relation. By colliding two operators on the Riemann surface together, we obtained one degenerate operator G. The partition function is now expressed as

$$Z \to \int \mathrm{d}a \, \langle G| \, a\rangle\langle a \, |G\rangle . \tag{12.13}$$

However, the naive decoupling limit $m \to 0$ kills the weight of the integration, i.e. we are integrating zero over an infinite parameter region. We need to properly define the limitation. On the other hand, however, we can take out just one piece of the integrand, with proper normalization, (?)

$$\langle G| \, a\rangle\langle a \, |G\rangle = Z_{instanton}. \tag{12.14}$$

13 Quantum Seiberg-Witten Curve and Resurgent Quantum Mechanics