Orbits of Subgroup Actions on Homogeneous Spaces

Jinpeng An

Peking University

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Basic Setting

- Let G be a connected real Lie group,
- $H, \Gamma \subset G$ be closed subgroups,
- $X = G/\Gamma$.
- Consider the translation action $H \curvearrowright X$: $H \times X \to X,\ (h, x) \mapsto hx$.

To obtain nontrivial dynamical properties of the H-action, assume

- G and H are noncompact,
- Γ is a lattice, namely, a discrete subgroup such that X carries a finite G-invariant measure μ_X.
- Example: $G = \mathrm{SL}_n(\mathbb{R}),\ \Gamma = \mathrm{SL}_n(\mathbb{Z}),\ X_n := \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$.

Basic questions:

(1) Study properties of H-orbits in X.

(2) Study properties of H-invariant measures on X.

The Space $X_n = \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$

We mainly concentrate on Question (1) for X_n (although many results mentioned below remain true in more general cases).

$X_n \cong$ the space of unimodular lattices in $\mathbb{R}^n$:

- A lattice $\Lambda \subset \mathbb{R}^n$ is called unimodular if $\mathrm{vol}(\mathbb{R}^n/\Lambda) = 1$.
- Example: $\Lambda = \mathbb{Z}^n$.
- $\mathrm{SL}_n(\mathbb{R})$ acts transitively on the space of unimodular lattices in $\mathbb{R}^n$, the stabilizer at $\mathbb{Z}^n$ is $\mathrm{SL}_n(\mathbb{Z})$.
- Identify $g\mathrm{SL}_n(\mathbb{Z}) \in X_n$ with $g\mathbb{Z}^n$.

X_n is noncompact. A subset $S \subset X_n$ is called bounded if $\overline{S}$ is compact.

Mahler's Compactness Criterion

$S \subset X_n$ is bounded iff 0 is an isolated point of $\bigcup_{\Lambda \in S} \Lambda$.

Almost Every Point Has Dense H-orbit

Moore's Ergodicity Theorem (1966)

Let $H \subset \mathrm{SL}_n(\mathbb{R})$ be a noncompact closed subgroup. Then the action $H \curvearrowright X_n$ is ergodic, i.e., for any H-invariant measurable subset $S \subset X_n$, one has

$$\mu_{X_n}(S) = 0 \quad \textit{or} \quad \mu_{X_n}(X_n \smallsetminus S) = 0.$$

This implies that for such H,

$$\mu_{X_n}(\{x \in X_n : \overline{Hx} \neq X_n\}) = 0.$$

However, non-dense H-orbits are important:

- They reveal the complexity of the action $H \curvearrowright X_n$;
- They are related to number-theoretic questions.

Margulis' Proof of the Oppenheim Conjecture

Theorem (Margulis, 1986)

Every bounded $\mathrm{SO}(2,1)$*-orbit in* X_3 *is compact.*

This implies:

Corollary (Oppenheim Conjecture, 1929)

Let $n \geqslant 3$, $Q : \mathbb{R}^n \to \mathbb{R}$ *be a nondegenerate indefinite quadratic form. Assume* Q *is not a constant multiple of an integral quadratic form. Then*

$$\inf_{v \in \mathbb{Z}^n \smallsetminus \{0\}} |Q(v)| = 0.$$

The condition $n \geqslant 3$ is necessary:

$$|x^2 - (1+\sqrt{2})^2 y^2| \geqslant 1, \qquad \forall\, (x,y) \in \mathbb{Z}^2 \smallsetminus \{(0,0)\}.$$

Sketched Proof of the Corollary. It suffices to prove the $n = 3$ case.

- Let $Q_0(x, y, z) = x^2 + y^2 - z^2$. Then $Q = c(Q_0 \circ g)$ for some $c \in \mathbb{R}$ and $g \in \mathrm{SL}_3(\mathbb{R})$.
- The condition "$Q \asymp$ integral form" implies the $\mathrm{SO}(2, 1)$-orbit of $g\mathbb{Z}^3 \in X_3$ is noncompact.
- By Margulis' Theorem, $\mathrm{SO}(2, 1)(g\mathbb{Z}^3)$ is unbounded in X_3.
- By Mahler's Criterion, 0 is not isolated in $\mathrm{SO}(2, 1)g\mathbb{Z}^3 \subset \mathbb{R}^3$, namely, there exist $h_n \in \mathrm{SO}(2, 1)$ and $v_n \in \mathbb{Z}^3 \setminus \{0\}$ such that $h_n g v_n \to 0$.
- This implies $|Q(v_n)| = |cQ_0(gv_n)| = |cQ_0(h_n g v_n)| \to 0$.

Thus

$$\inf_{v \in \mathbb{Z}^n \setminus \{0\}} |Q(v)| = 0.$$

□

Ratner's Theorem

Ratner's Orbit Closure Theorem (1991)

Let $H \subset \mathrm{SL}_n(\mathbb{R})$ be a connected closed subgroup that is generated by unipotent one-parameter subgroups. Then for every $x \in X_n$, there exists a connected closed subgroup $L \subset \mathrm{SL}_n(\mathbb{R})$ with $L \supset H$ s.t. $\overline{Hx} = Lx$.

Examples of H:

- Every $h \in H$ is upper triangular with 1's on the diagonal.
- H is noncompact simple (by Iwasawa decomposition).

Proof of Margulis' Theorem from Ratner's Theorem. Let $x \in X_3$ be such that $\mathrm{SO}(2,1)x$ is bounded.

- By Ratner's Theorem, there exists a connected closed subgroup $L \subset \mathrm{SL}_3(\mathbb{R})$ containing $\mathrm{SO}^+(2,1)$ such that $\overline{\mathrm{SO}^+(2,1)x} = Lx$.
- $\mathrm{SO}^+(2,1)$ is a maximal connected proper subgroup of $\mathrm{SL}_3(\mathbb{R})$.
- "$\mathrm{SO}(2,1)x$ is bounded" $\implies L \neq \mathrm{SL}_3(\mathbb{R}) \implies L = \mathrm{SO}^+(2,1)$ $\implies$ "$\mathrm{SO}^+(2,1)x$ is closed" $\implies$ "$\mathrm{SO}(2,1)x$ is compact". □

Conjectures of Cassels-Swinnerton-Dyer and Margulis

For subgroups without nontrivial unipotent elements, Margulis stated:

Conjecture (Margulis, 2000)

Let $n \geqslant 3$, $H = \{\mathrm{diag}(h_1, \ldots, h_n) : h_i > 0, h_1 \cdots h_n = 1\}$. Then every bounded H-orbit in X_n is compact.

This is equivalent to (by Mahler's Criterion):

Conjecture (Cassels-Swinnerton-Dyer, 1955)

Let $n \geqslant 3$, $\{f_1, \ldots, f_n\}$ be a basis of $(\mathbb{R}^n)^*$, and $F = f_1 \cdots f_n$. Assume F is not a constant multiple of an integral polynomial. Then

$$\inf_{v \in \mathbb{Z}^n \setminus \{0\}} |F(v)| = 0.$$

For $n = 2$, both statements are not true.

Littlewood's Conjecture

The $n = 3$ case of the above conjectures imply:

Littlewood's Conjecture (1930s)

For any $a, b \in \mathbb{R}$, one has

$$\inf_{m \in \mathbb{N}} m \cdot \mathrm{dist}(ma, \mathbb{Z}) \cdot \mathrm{dist}(mb, \mathbb{Z}) = 0.$$

Relation to H-action:

- For $a, b \in \mathbb{R}$, let $x_{a,b} = \begin{pmatrix} 1 & & a \\ & 1 & b \\ & & 1 \end{pmatrix} \mathrm{SL}_3(\mathbb{Z}) \in X_3$.
- Let $H^+ = \left\{ \begin{pmatrix} h_1 & & \\ & h_2 & \\ & & (h_1 h_2)^{-1} \end{pmatrix} : h_1, h_2 \geqslant 1 \right\}$.
- Mahler's Criterion implies: $\inf_{m \in \mathbb{N}} m \cdot \mathrm{dist}(ma, \mathbb{Z}) \cdot \mathrm{dist}(mb, \mathbb{Z}) = 0$ iff $H^+ x_{a,b}$ is unbounded.

Work of Einsiedler-Katok-Lindenstrauss

Theorem (Einsiedler-Katok-Lindenstrauss, 2006)

Let $n \geqslant 3$, $H = \{\mathrm{diag}(h_1, \ldots, h_n) : h_i > 0, h_1 \cdots h_n = 1\}$. *Then*

$$\dim_H\{x \in X_n : Hx \textit{ is bounded}\} = n - 1.$$

Remark: There are only countably many compact H-orbits in X_n. So

$$\dim_H\{x \in X_n : Hx \text{ is compact}\} = n - 1.$$

A stronger form of the above theorem implies:

Corollary (Einsiedler-Katok-Lindenstrauss, 2006)

Littlewood's Conjecture holds up to a set of Hausdorff dimension 0, *i.e.,*

$$\dim_H \left\{(a, b) \in \mathbb{R}^2 : \inf_{m \in \mathbb{N}} m \cdot \mathrm{dist}(ma, \mathbb{Z}) \cdot \mathrm{dist}(mb, \mathbb{Z}) > 0\right\} = 0.$$

One-parameter Subgroup Actions: Bounded Orbits

Let $H = \{\exp(t\xi) : t \in \mathbb{R}\} \subset \mathrm{SL}_n(\mathbb{R})$, where $\xi \in \mathfrak{sl}_n(\mathbb{R})$.

Theorem (Ratner, 1991)

If H is unipotent, then the set $\{x \in X_n : \overline{Hx} \neq X_n\}$ is contained in a countable union of proper submanifolds of X_n.

Theorem (Kleinbock-Margulis, 1996)

If H is diagonalizable, then

$$\dim_H \{x \in X_n : Hx \text{ is bounded}\} = \dim X_n.$$

Conjecture (A.-Guan-Kleinbock, 2015)

Let $H_1, H_2, \ldots$ be countably many diagonalizable one-parameter subgroups of $\mathrm{SL}_n(\mathbb{R})$. *Then*

$$\dim_H \{x \in X_n : \text{all } H_k x \text{ are bounded}\} = \dim X_n.$$

- Motivation: Schmidt's Conjecture in Diophantine approximation.
- The $n = 2$ case is known.

Theorem (A.-Guan-Kleinbock, 2015)

The conjecture holds for $n = 3$.

For arbitrary n, partial result is proved by Guan-Wu (2018).

One-parameter Subgroup Actions: Divergent Forward Orbits

- Let $H^+ = \{\exp(t\xi) : t \geqslant 0\} \subset \mathrm{SL}_n(\mathbb{R})$.
- For $x \in X_n$, H^+x is divergent if for any compact subset $K \subset X_n$, there exists $t_K > 0$ such that "$t > t_K$" $\Longrightarrow$ "$\exp(t\xi)x \notin K$".
- Denote $D_\xi(X_n) = \{x \in X_n : H^+x \text{ is divergent}\}$.
- Moore's Ergodicity Theorem $\Longrightarrow \mu_{X_n}(D_\xi(X_n)) = 0$.

Theorem (Margulis, 1971)

If H is unipotent, then $D_\xi(X_n) = \varnothing$.

Theorem (Guan-Shi, 2020)

In general, $\dim_H D_\xi(X_n) < \dim X_n$.

Example: Let $n = 2$, $\xi = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathrm{SL}_2(\mathbb{Z}) \in D_\xi(X_2)$ iff a and b are linearly dependent over $\mathbb{Q}$. Thus $\dim_H D_\xi(X_2) = 2$.

In contrast, one has:

Theorem (Das-Fishman-Simmons-Urbánski, 2019+)

Let $n \geqslant 3$, and $p, q \geqslant 1$ with $p + q = n$. Let $\xi_{p,q} = \begin{pmatrix} \frac{1}{p} I_p & \\ & -\frac{1}{q} I_q \end{pmatrix}$. Then

$$\dim_H D_{\xi_{p,q}}(X_n) = \dim X_n - \frac{pq}{n}.$$

- Partial results known before:
  - (Cheung 2011): $n = 3$.
  - (Cheung-Chevallier 2016): $p = 1$.
  - (Kadyrov-Kleinbock-Lindenstrauss-Margulis 2017): "$\leqslant$".
- This is related to singular matrices in Diophantine approximation. DFSU's theorem is equivalent to: $\dim_H \mathrm{Sing}(p, q) = pq(1 - \frac{1}{p+q})$.

One-parameter Subgroup Actions: Non-dense Orbits

Let $H \subset \mathrm{SL}_n(\mathbb{R})$ be a diagonalizable one-parameter subgroup.

Question

Let $S \subset X_n$ be a subset. Under what conditions on S does one have

$$\dim_H \{x \in X_n : \overline{Hx} \cap S = \varnothing\} = \dim X_n?$$

Theorem

This holds if one of the following conditions hold:

(1) *S is H-invariant, closed, and has measure* 0 (Kleinbock-Margulis, 1996);

(2) *S is finite* (Kleinbock, 1998);

(3) *S is countable* (A.-Guan-Kleinbock, 2020+).

A more general version of (3) implies:

Theorem (A.-Guan-Kleinbock, 2020+)

Let M be a Riemannian locally symmetric space of noncompact type, $S \subset M$ be a countable subset, and $x \in M \smallsetminus S$. Then

$$\dim_H \{\ell \in \mathbb{P}(T_x M) : \overline{\exp_x(\ell)} \cap S = \varnothing\} = \dim M - 1.$$

Relation to Diophantine Approximation

Dirichlet's Approximation Theorem for Matrices

For any $A \in \mathrm{M}_{p\times q}(\mathbb{R})$ *and* $Q \in \mathbb{R}_{\geqslant 1}$, *there exists* $v \in \mathbb{Z}^q \smallsetminus \{0\}$ *with* $\|v\|_\infty \leqslant Q$ *such that* $\mathrm{dist}_\infty(Av, \mathbb{Z}^p) < Q^{-q/p}$. *In particular, there exist infinitely many* $v \in \mathbb{Z}^q \smallsetminus \{0\}$ *such that* $\|v\|_\infty^{q/p} \cdot \mathrm{dist}_\infty(Av, \mathbb{Z}^p) < 1$.

Definition

Let $A \in \mathrm{M}_{p\times q}(\mathbb{R})$.

- A is Dirichlet improvable if there exist $\epsilon \in (0, 1)$ and $Q_0 \geqslant 1$ such that for any $Q \geqslant Q_0$, there exists $v \in \mathbb{Z}^q \smallsetminus \{0\}$ with $\|v\|_\infty \leqslant Q$ such that $\mathrm{dist}_\infty(Av, \mathbb{Z}^p) < \epsilon Q^{-q/p}$.
- A is singular if for any $\epsilon \in (0, 1)$, there exists $Q_0 \geqslant 1$ such that for any $Q \geqslant Q_0$, there exists $v \in \mathbb{Z}^q \smallsetminus \{0\}$ with $\|v\|_\infty \leqslant Q$ such that $\mathrm{dist}_\infty(Av, \mathbb{Z}^p) < \epsilon Q^{-q/p}$.
- A is badly approximable if $\inf_{v \in \mathbb{Z}^q \smallsetminus \{0\}} \|v\|_\infty^{q/p} \cdot \mathrm{dist}_\infty(Av, \mathbb{Z}^p) > 0$.

Denote

- $\mathrm{DI}(p,q) = \{A \in \mathrm{M}_{p\times q}(\mathbb{R}) : A$ is Dirichlet improvable$\}$,
- $\mathrm{Sing}(p,q) = \{A \in \mathrm{M}_{p\times q}(\mathbb{R}) : A$ is singular$\}$,
- $\mathrm{Bad}(p,q) = \{A \in \mathrm{M}_{p\times q}(\mathbb{R}) : A$ is badly approximable$\}$.

It is know that

- $\mathrm{DI}(p,q) \supset \mathrm{Sing}(p,q) \cup \mathrm{Bad}(p,q)$.
- $\mathrm{DI}(1,1) = \mathrm{Sing}(1,1) \cup \mathrm{Bad}(1,1),\ \mathrm{Sing}(1,1) = \mathbb{Q}$.
- $\mathrm{Leb}\big(\mathrm{DI}(p,q)\big) = 0$.
- $\dim_H \mathrm{Bad}(p,q) = pq$ (Schmidt, 1969).
- $\dim_H \mathrm{Sing}(p,q) = pq(1 - \frac{1}{p+q})$ for $(p,q) \neq (1,1)$ (Das-Fishman-Simmons-Urbánski, 2019+).

- Let $n = p + q$.
- For $A \in \mathrm{M}_{p\times q}(\mathbb{R})$, let $x_A = \begin{pmatrix} I_p & A \\ 0 & I_q \end{pmatrix} \mathrm{SL}_n(\mathbb{Z}) \in X_n$.
- Let $H^+ = \left\{ h_t := \begin{pmatrix} e^{\frac{t}{p}} I_p & \\ & e^{-\frac{t}{q}} I_q \end{pmatrix} : t \geqslant 0 \right\} \subset \mathrm{SL}_n(\mathbb{R})$.

Mahler's Criterion implies:

Dani Correspondence

For $A \in \mathrm{M}_{p\times q}(\mathbb{R})$,

- $A \in \mathrm{Bad}(p,q)$ *iff* $H^+ x_A$ *is bounded;*
- $A \in \mathrm{Sing}(p,q)$ *iff* $H^+ x_A$ *is divergent;*
- $A \in \mathrm{DI}(p,q)$ *iff* $\omega(x_A) \cap S = \varnothing$, *where*

$$\omega(x_A) = \{y \in X_n : \exists\, t_k \to +\infty \text{ s.t. } h_{t_k} x_A \to y\},$$

$$S = \bigcup_{\sigma \in S_n} U_\sigma x_0, \text{ each } U_\sigma \text{ a maximal unipotent subgroup of } \mathrm{SL}_n(\mathbb{R}).$$

THANK YOU !