Classical groups and their representations - an introduction

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SUDA Lie Group Lecture Series

(November 3, 2020)

1 A group should be understood by its actions

- One can tell a lot on the real nature of a group by knowing what possible places it may appear as a group of transformations, namely through its actions.
- Out of all actions, linear actions (by invertible linear transformations) are by far the simplest.
  - They are (called) representations.

- All actions can be converted in some sense to linear actions by the following scheme:
  - If $G \curvearrowright X$, let

$$\boxed{C(X) = \text{ space of functions on } X.}$$

(think $C(X)$ as the full collection of observables on X)

Then G acts on $C(X)$ by:

$$(g \cdot F)(x) = F(g^{-1} \cdot x), \quad g \in G.$$

  - Other vector spaces may also be considered, including
    * $L^2(X)$ (if there is a suitable measure on X),
    * space of sections of a vector bundle on X, and
    * various cohomological spaces on X .

- To understand a linear operator, one should perform spectral analysis (i.e., eigenspace decomposition).
- To understand a representation, one should complete the same task:
  - analogue of an eigenspace: <u>irreducible representation</u>
  - harmonic analysis: decompose a given representation into irreducible components.

2 The orthogonal group and its natural representation

The (compact and pseudo) orthogonal groups

- $O(m)$: the group of linear transformations on $\mathbb{R}^m$ preserving the distance (squared)

$$x_1^2 + \cdots + x_m^2.$$

- More generally $O(p, q)$: the group of linear transformations on $\mathbb{R}^m$ preserving the “pseudo distance”

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_m^2,$$

where $p + q = m$.

Remarks.

- $\mathbb{R}^m$ is where the orthogonal group $O(m)$ resides:
  - called the natural representation
- $\mathbb{R}^m$ is more "basic" than $O(m)$:
  - $\dim \mathbb{R}^m = m$
  - $\dim O(m) = \frac{m(m-1)}{2}$
- We should strive to construct representations of $O(m)$ from the more basic entity $\mathbb{R}^m$.

We will focus on the natural action of $O(m)$ on $\mathbb{R}^m$.

- It divides $\mathbb{R}^m$ into orbits:

$$S^{m-1}(r) = \{x \in \mathbb{R}^m : x_1^2 + \cdots + x_m^2 = r^2\}, \quad r \geq 0.$$

- Each $S^{m-1}(r)$ is a homogeneous space for $O(m)$, and
-

$$S^{m-1}(r) \simeq S^{m-1}(r'), \quad r, r' > 0,$$

as $O(m)$ homogeneous spaces.

Take a typical orbit, say the unit sphere $S^{m-1} = S^{m-1}(1)$, and consider the corresponding representation of $O(m)$ on a space of functions on S^{m-1}, say $L^2(S^{m-1})$.

Theory of spherical harmonics:

- $L^2(S^{m-1}) = \sum_{k \in \mathbb{Z}_{\geq 0}} H_k$, where H_k consists of the restrictions to S^{m-1} of harmonic polynomial functions on $\mathbb{R}^m$ of total degree k.
- H_k may also be characterized as the eigenspace of the Laplace-Beltrami operator of eigenvalue $-k(m+k-2)$.
- The spaces H_k are all irreducible under $O(m)$.

Actually, it is better to consider the representation of $O(m)$ on $L^2(\mathbb{R}^m)$:

- The space $L^2(\mathbb{R}^m)$ allows more symmetries (such as dilation, Fourier transform).
- Theory of spherical harmonics is a part of the spectral decomposition of $L^2(\mathbb{R}^m)$ as a representation of $O(m)$.

More precisely, we have the <u>isotypic decomposition</u>

$$L^2(\mathbb{R}^m) = \sum_{\lambda \in \mathrm{Irr}(O(m))} L^2(\mathbb{R}^m)_\lambda = \sum_{\lambda \in \mathrm{Irr}(O(m))} L^2(\mathbb{R}^m; \lambda') \otimes V_\lambda.$$

- $L^2(\mathbb{R}^m; \lambda')$ is the space of multiplicities of λ, which carries additional symmetries.
- $L^2(\mathbb{R}^m; \lambda') \neq 0$ if and only if $\lambda \simeq H_k$, for some $k \in \mathbb{Z}_{\geq 0}$.

3 Going beyond the natural representation

- An obvious way to go beyond the natural representation of $O(m)$ on $\mathbb{R}^m$ is to consider the direct sum of (say) n copies of $\mathbb{R}^m$, which is $M_{m,n}(\mathbb{R})$, the space of $m \times n$ real matrices.
- Thus $O(m)$ acts on $M_{m,n}(\mathbb{R})$, now by matrix multiplication on the left.
- As in $L^2(\mathbb{R}^m)$, we now consider $L^2(M_{m,n}(\mathbb{R}))$.

- Now in $M_{m,n}(\mathbb{R})$, there is more "space" for $O(m)$ to move around, and hence $L^2(M_{m,n}(\mathbb{R}))$ can "accommodate" more representations of $O(m)$.
- In fact all irreducible representations of $O(m)$ will appear in $L^2(M_{m,n}(\mathbb{R}))$ as soon as $n \geq m$.

4 From compact orthogonal groups to pseudo orthogonal groups

- Consider the pseudo orthogonal group $O(p,q)$ and its natural action on $\mathbb{R}^m$, where $m = p + q$. The inner product is

$$(x,x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_m^2.$$

  - The orbits are of the form $\boxed{(x,x) = t}$: four different types.
    - roughly depending on the sign of t;
    - two orbits for $t = 0$ (cone minus the origin, and the origin).
  - This leads to a more complicated structure of $L^2(\mathbb{R}^m)$ as a representation of $O(p,q)$.

- Representations of $O(p,q)$ on these orbits are all part of spectral analysis of $L^2(\mathbb{R}^m)$.
- Similarly for the $O(p,q)$ representation on $L^2(M_{m,n}(\mathbb{R}))$.
- Due to the non-compact nature of $O(p,q)$, many issues arise:
  - finite dimensional vs infinite dimensional;
  - unitary versus non-unitary.
- We shall consider all (smooth) representations.

5 A basic idea in spectral analysis

- An important way to decompose a representation is to find operators which commute with the group action.
  - the commuting or intertwining algebra.
- A very good scenario is when the commuting algebra comes from a group action.

- An even better scenario is when they are mutual centralizers under a larger group action.
- This is indeed the case for the $O(p,q)$ representation $L^2(M_{m,n}(\mathbb{R}))$: there is a commuting action of the metaplectic group $\widetilde{Sp}(2n,\mathbb{R})$ on $L^2(M_{m,n}(\mathbb{R}))$, which are the

$$\boxed{\text{“hidden” symmetries}}.$$

Example: extra symmetry by an $\widetilde{SL}(2,\mathbb{R})$ action on $L^2(\mathbb{R}^m)$.

- The representation is called an oscillator representation, denoted by ω.
- We describe the representation ω by generators of $SL(2,\mathbb{R})$:

$$m_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},\ n_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},\ \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$$(\omega(m_a)f)(x) = |a|^{\frac{p+q}{2}} f(ax),$$

(normalized dilation)

$$(\omega(n_b)f)(x) = e^{\frac{ib}{2}(x,x)} f(x),$$

(multiplication by an $O(p,q)$-invariant function)

$$(\omega(\sigma)f)(x) = (\frac{1}{2\pi})^{\frac{p+q}{2}} \int_{R^m} e^{-i(x,y)} f(y)dy.$$

(Fourier transform)

6 An remarkable phenomenon: Howe duality

- Consider the space $\mathcal{S}(M_{m,n}(\mathbb{R}))$ of Schwartz class functions on $M_{m,n}(\mathbb{R})$.
- This is a representation of $O(p, q) \times \widetilde{Sp}(2n, \mathbb{R})$.
  - $O(p, q)$ acts naturally.
  - $\widetilde{Sp}(2n, \mathbb{R})$ acts by "hidden symmetries".
- Denote this representation by $\omega_{m,n}$: the smooth oscillator representation.

- We consider quotient representations of $\omega_{m,n}$.
- Questions:
  - What representation π of $O(p, q)$ may appear as a quotient of $\omega_{m,n}$?
  - What representation σ of $\widetilde{Sp}(2n, \mathbb{R})$ may appear as a quotient of $\omega_{m,n}$?

Dual pair correspondence:

- Given $\pi \in \mathrm{Irr}(O(p,q))$ and $\sigma \in \mathrm{Irr}(\widetilde{Sp}(2n,\mathbb{R}))$, there is at most one way for $\pi \otimes \sigma$ to appear as $O(p,q) \times \widetilde{Sp}(2n,\mathbb{R})$-quotient of $\omega_{m,n}$.
- If π appears as a $O(p,q)$-quotient of $\omega_{m,n}$, there is a unique representation σ such that $\pi \otimes \sigma$ appears as $O(p,q) \times \widetilde{Sp}(2n,\mathbb{R})$-quotient of $\omega_{m,n}$, and likewise for σ.
- π and σ are said to correspond under $\omega_{m,n}$.

$$\begin{array}{ccc} & \omega_{m,n} & \\ \swarrow & & \searrow \\ \pi \in \mathrm{Irr}(O(p,q)) & & \sigma \in \mathrm{Irr}(\widetilde{Sp}(2n,\mathbb{R})) \end{array}$$

- The correspondence is defined by the condition

$$\mathrm{Hom}_{O(p,q)\times\widetilde{Sp}(2n,\mathbb{R})}(\omega_{m,n},\pi\otimes\sigma)\neq 0.$$

  – The correspondence is always $1 \leftrightarrow 1$.

- Other names: Howe correspondence or local theta correspondence

The general lessons:

- Representations of $O(p,q)$ arising from orbits in $M_{m,n}(\mathbb{R})$ are all part of the spectral analysis of $\omega_{m,n}$.
- Representations of $O(p,q)$ (occurring in $\omega_{m,n}$) should be understood together with, and through representations of $\widetilde{Sp}(2n,\mathbb{R})$ (occurring in $\omega_{m,n}$).

7 A fundamental issue: occurrence

- What is the domain of the correspondence for $\omega_{m,n}$?
- How does one detect occurrence?

Persistence: (Kudla)

- if $\pi \in \mathrm{Irr}(O(p,q))$ occurs in the duality correspondence with $\widetilde{Sp}(2n,\mathbb{R})$, then it occurs in the duality correspondence with $\widetilde{Sp}(2n+2l,\mathbb{R})$, for $l \geq 0$.
- if $\sigma \in \mathrm{Irr}(\widetilde{Sp}(2n,\mathbb{R}))$ occurs in the duality correspondence with $O(p,q)$, then it occurs in the duality correspondence with $O(p+l,q+l)$, for $l \geq 0$.

Thus "**Once occur, forever occur**" (along a Witt tower).

Stable range occurrence: (Howe)

- Every representation π of $O(p,q)$ occurs in the duality correspondence with $\widetilde{Sp}(2n,\mathbb{R})$, if $p+q\le n$.
- Every (genuine) representation σ of $\widetilde{Sp}(2n,\mathbb{R})$ occurs in the duality correspondence with $O(p,q)$, if $p,q\ge 2n$.

Terminology:

- The dual pair (G,G') is in the stable range, with G the small member.

Notation:

$$G\le \frac{G'}{2}$$

(All representations of G then occur in the dual pair correspondence)

- First occurrence (along a Witt tower) thus carries critical information.
- Kudla-Rallis conjectured certain conservation relations on the first occurrence indices: (mid 1990's)
  - Some particular cases: by Kudla-Rallis and others
  - Established in full generality by Sun-Zhu (JAMS 2015).

- Conservation relation for the orthogonal group:

$$\mathrm{n}(\pi) + \mathrm{n}(\pi \otimes \det) = p + q, \quad \forall \pi \in \mathrm{Irr}(O(p,q)).$$

where

$$\mathrm{n}(\pi) := \min\{n \,|\, \pi \text{ occurs in the correspondence with } \widetilde{Sp}(2n,\mathbb{R})\}.$$

- Early occurrence of one implies late occurrence of the other.

- Examples: (the four characters)
  - $\mathrm{n}(\mathbf{1}) = 0$, $\mathrm{n}(\det) = p + q$;
  - $\mathrm{n}(\mathbf{1}^{+,-}) = p$, $\mathrm{n}(\mathbf{1}^{-,+}) = q$.
- Meaning of $\mathrm{n}(\det) = p + q$:
  - $\mathcal{S}^\star(M_{p+q,n}(\mathbb{R}))^{O(p,q),\,\det} = 0$, if $n < p + q$.
  - $\mathcal{S}^\star(M_{p+q,n}(\mathbb{R}))^{O(p,q),\,\det} \neq 0$, if $n = p + q$.

8 The key task

- Describe the correspondence in terms of the Langlands parameters.
  - Lots of works have been done; not yet complete.
- Understand the correspondence, in terms of invariants of representations.
  - Qualitative information: e.g. infinitesimal character, nilpotent invariants

9 Applications to unitary dual

- Preservation of unitarity in the stable range (J.-S. Li):
  - If $G \leq \frac{G'}{2}$, then all unitary representations of G lift to unitary representations of G'.
- The resulting representations of G' are called singular unitary representations, an important yet still mysterious part of the unitary dual.

- The most mysterious unitary representations are the so-called "unipotent" representations:
  - They are "associated" to nilpotent orbits, in the orbit philosophy of Kirillov and Kostant.
- Parts of them are related to Langlands philosophy, called special unipotent representations (Arthur, Barbasch-Vogan).
  - Barbasch-Ma-Sun-Zhu (ongoing): construction and classification of special unipotent representations;
  - Heavily using theory of local theta correspondence.

10 Concluding messages

- All representations of the orthogonal group $O(p,q)$ can be found by studying various function spaces on $M_{p+q,n}(\mathbb{R})$.
- Representations of $O(p,q)$ should be studied together with representations of $\widetilde{Sp}(2n,\mathbb{R})$, for all n.

Thank you !